Causal Reconstruction Kernels for Consistent Signal Recovery

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Motivation – Causal Reconstruction

Causal signal reconstruction is crucial

- in on-line applications
- for feedback loops in control systems
- to reduce border effects in image processing, etc.

Framework – Shift Invariant Sampling

Non-ideal Acquisition

- Impulse filter
- Linear filter

\[ x_c(t) = y(na) = \int_{-\infty}^{\infty} h(\tau) x(na - \tau) d\tau \]

- Sampling functions \( s_n \in \mathcal{H} \) have the form \( s_n = (T^n s)(t) = s(t - na) \)
- or in general \( s_n = U^ns \) with \( U \) unitary operator on \( \mathcal{H} \)

\[ \mathcal{S} = \{ s_n \}_{n \in \mathbb{Z}} \] of sampling functions forms a stationary sequence in \( \mathcal{S} \)

\[ \mathcal{S} \ni s \text{ is a Riesz basis for the sampling space } \mathcal{S} \]

\[ \Rightarrow \mathcal{S} = \{ s_n \}_{n \in \mathbb{Z}} \] of sampling functions forms a stationary sequence in \( \mathcal{H} \)

\[ \mathcal{S} \ni s \text{ is characterized by its corresponding spectral density } \Phi_s(e^{i\theta}) \]

Example – Causal Spline Reconstruction

Practical Assumptions

- Shift-invariant sampling in \( L^2(\mathbb{R}) \) with period \( a = 1 \):
  \( U^n s(t) = s(t - na) \)
- Impulse response \( s(t) \) is a B-spline of 2nd degree as a model of a non-ideal lowpass

Causal versus Non-causal Reconstruction Kernels

- Dashed lines: truncated non-causal reconstruction kernels
- Solid lines: causal reconstruction kernels

Signal Reconstruction – Causal versus Non-causal

- Significant differences close to the border
- Causal and non-causal reconstruction coincide at the distant past \( t \to -\infty \)

Main Result – Causal Dual Basis

Theorem

Let \( s = \{ s_n \}_{n \in \mathbb{Z}} \) be a stationary sequence in a Hilbert space \( \mathcal{H} \) which is a Riesz basis for \( \mathcal{S} = \{ S_n \}_{n \in \mathbb{Z}} \) and let \( \Phi_s \) be the spectral density of \( s \).

Then \( s_n = \{ s_n \}_{n \in \mathbb{Z}} \) is a Riesz basis for \( \mathcal{S} = \{ S_n \}_{n \in \mathbb{Z}} \) and the corresponding dual Riesz basis \( \{ \zeta_n \}_{n \in \mathbb{N}} \) is given by

\[ \zeta_n = \sum_{k=-\infty}^{k=\infty} \hat{\varphi}_n(k) s_{n+k}, \quad n = 0, 1, 2, \ldots \]

with

\[ \hat{\varphi}_n(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi_n(e^{i\theta}) e^{-ik\theta} d\theta, \quad n = 0, 1, 2, \ldots \]

and wherein \( \Phi_s^* \) and \( \Phi_s \) are the spectral factors of \( \varphi_s \).

Reconstructed past signal

\[ \tilde{x}_n(t) = \sum_{n=-\infty}^{\infty} c_n s_n, \quad t \leq t_0 \]

Each reconstruction kernel \( \zeta_n(t) \) has a different shape

Each sampling value corresponds to the weighting of the respective kernel

Reconstruction – Ideal World

Assumptions

- Let \( x \in \mathcal{S} \) be an arbitrary signal
- All past and future signal samples \( c_n = (x, s_n), n \in \mathbb{Z} \) are known

Goal

Signal reconstruction of the form

\[ \tilde{x}(t) = \sum_{n=-\infty}^{\infty} (x, s_n) \sigma_n(t) \]

such that

\[ \tilde{x}(t) = x(t) \quad \text{for all } x \in \mathcal{S} \] (Perfect reconstruction)

\[ \tilde{x}(s_n) = (x, s_n) \quad \text{for all } n \in \mathbb{Z} \] (Consistency)

Solution

A well known result from frame theory states that the problem is solved by the dual Riesz basis \( \{ \sigma_n \}_{n \in \mathbb{Z}} \) of \( \{ s_n \}_{n \in \mathbb{Z}} \). Given by

\[ \sigma_n(t) = (U^n s)(t) \quad \text{and} \quad \sigma_n(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{\varphi}_n(e^{i\theta}) e^{ik\theta} d\theta \]

Often, in reality only the past signal samples are known!

Reconstruction – Real World

New Assumption

Let \( c_0 = (x_0, s_\cdots) \) be the past signal samples known at \( t = t_0 \)

Goal

Reconstruct the past signal component

\[ x_\cdots(t) = \begin{cases} x(t) & \text{if } t \leq t_0 \\ 0 & \text{if } t > t_0 \end{cases} \]

of the signal \( x(t) \) at time \( t = t_0 \) from the past signal samples \( s_\cdots \) only.

Naive Solution

\[ \tilde{x}_\cdots(t) = \sum_{n=-\infty}^{n=t_0} c_n \sigma_{n-t_0}(t), \quad t \leq t_0 \]

based on the non-causal dual frame (1).

However this reconstruction is not perfect, i.e. \( \tilde{x}_\cdots \neq x_\cdots \), because we need the dual Riesz basis \( \{ \zeta_n \}_{n \in \mathbb{N}} \)!