Phase Retrieval via Structured Modulations In Paley Wiener Spaces

Fanny Yang, Volker Pohl, Holger Boche

Institute for Theoretical Information Technology
Technische Universität München

SampTA 2013, Jacobs University Bremen
July 4th, 2013
Overview

- Phase Retrieval
  - Applications
  - Finite dimensional

- Main result – Paley-Wiener spaces
  - Measurement setup
  - Choice of measurement parameters

- Corollary

- Discussion and outlook
X-Ray crystallography

\[ x \rightarrow |\mathcal{F}\{x\}| \rightarrow \text{Extract } x \]

\( P \rightarrow \) Intensity detector

\( \text{monochromatic X-rays} \rightarrow \) crystal

Phase retrieval problem
Phase Retrieval – K dimensions

Signal processing approach: K-dimensional

| Problem formulation | Given \( c_m = |\langle a_m, x \rangle|^2 \) for \( m = 1, \ldots, M \) | Recover \( x \in \mathbb{C}^K \) |
|---------------------|-------------------------------------------------|----------------------|

→ Determine \( a_m \) and \( M \)

When is \( x \mapsto \langle a_m, x \rangle \) injective?

- Choose \( \{a_m\} \) as basis
- \( M = K \) (e.g. DFT)

When is \( x \mapsto |\langle a_m, x \rangle|^2 \) injective?

- Generic frame: \( M \geq 4K-2 \) (or 4K-4?)
- Choose \( \{a_m\} \) as MUB or as a 2-uniform M/K tight frame with \( M = K^2 \)

---

Phase Retrieval - Motivation

K dimensions $\rightarrow \infty$ dimensions?
Phase Retrieval — $\infty$-dimensional signal space

Time (spatially) limited signals $x \in \mathcal{L}^2(\mathbb{T})$ with $\mathbb{T} = [-\frac{T}{2}, \frac{T}{2}]$

Define „Fourier Transform“: $x \mapsto \hat{x}(z) := \int_{\mathbb{T}} x(t) e^{itz} \, dt, \forall z \in \mathbb{C}$

$\rightarrow$ $x \in \mathcal{L}^2(\mathbb{T}) \iff \hat{x}(z) \in \mathcal{PW}_{T/2}$

- $\hat{x}(z)$ is an entire function with $|\hat{x}(z)| \leq C e^{\frac{T}{2}|z|}$
- $\int_{\mathbb{R}} |\hat{x}(\omega)|^2 d\omega < \infty$

$\rightarrow$ Isomorphic with inverse:

$$x(t) = \frac{1}{T} \int_{\mathbb{R}} \hat{x}(\omega) e^{it\omega} \, d\omega, \forall t \in \mathbb{T}$$

Recovery of $x(t) \leftrightarrow$ Reconstruction of $\hat{x}(z)$

Goal: Find measurement scheme and thus $\phi_n^{(m)}(\cdot)$ leading to the subproblems:

1. Finite dimensional phase retrieval
2. Interpolation from samples

Phase Retrieval – Approach overview

Given $c_n^{(m)} = |\phi_n^{(m)}(\hat{x})|^2$ for $m = 1, \ldots, M, \; n \in \mathbb{Z}$
Recover $\hat{x} \in \mathcal{PW}_{T/2}$

with linear measurement functional $\phi_n^{(m)}(\cdot) : \mathcal{PW}_{T/2} \to \mathbb{C}$ determined by specific measurement setup

$x \mapsto |\langle a_m, x \rangle|^2 \quad \Rightarrow \quad \hat{x} \mapsto |\phi_n^{(m)}(\hat{x})|^2$
Our approach – Measurement setup

Setup Parameters      Reconstruction

Modulation

Intensity measurement

Sampling with rate $1/\beta$

$$\begin{align*}
x(t) & \quad \rightarrow \quad y^{(1)}(t) \\
p^{(1)}(t) & \quad \rightarrow \quad \hat{y}^{(1)}(t) \\
\cdots & \quad \rightarrow \quad \cdots \\
p^{(M)}(t) & \quad \rightarrow \quad \hat{y}^{(M)}(t) \\
\rightarrow & \quad \rightarrow \\
c^{(1)}_n & = |\hat{y}^{(1)}(n\beta)|^2 \\
\cdots & = |\hat{y}^{(M)}(n\beta)|^2
\end{align*}$$

Motivation  Main results  Corollary  Discussion

$$p^{(m)}(t) := \sum_{k=1}^{K} \alpha_k^{(m)} e^{i \tilde{\lambda}_k t}, \quad m = 1, \ldots, M, \quad \tilde{\lambda}_k \in \mathbb{C}$$

$$\alpha^{(m)} := \left( \begin{array}{c} \alpha_1^{(m)} \\ \vdots \\ \alpha_K^{(m)} \end{array} \right), \quad \hat{x}_n := \left( \begin{array}{c} \hat{x}(n\beta + \tilde{\lambda}_1) \\ \vdots \\ \hat{x}(n\beta + \tilde{\lambda}_K) \end{array} \right)$$

$$c^{(m)}_n := \left| \sum_{k=1}^{K} \alpha_k^{(m)} \hat{x}(n\beta + \tilde{\lambda}_k) \right|^2 = \left| \langle \alpha^{(m)}, \hat{x}_n \rangle_{\mathbb{C}^K} \right|^2$$

For fix $n$ as in K-dimensional phase retrieval!
Choice of modulator coefficients \( \alpha^{(m)} \)

1. For each \( n \) solve
   \[
   \left\{ |\langle \alpha^{(m)}, \hat{x}_n \rangle|^2 \right\}_{m=1, \ldots, M} \to \hat{x}_n
   \]

   - Choice of vectors \( \alpha^{(m)} \)? How big is \( M \)?
   - Sufficient condition: \( \{ \alpha^{(m)} \} \) is 2-uniform \( M/K \)-tight, \( M = K^2 \)

   Instead of \( \hat{x}_n \) obtain \( Q_{\hat{x}_n} := \hat{x}_n \hat{x}_n^* \)

\[
Q_{\hat{x}_n} = \frac{(K + 1)}{K} \sum_{m=1}^{M} c_n^{(m)} Q_{\alpha^{(m)}} - \frac{1}{K} \sum_{m=1}^{M} c_n^{(m)} I
\]

A

\[
\min_{Q_x} \log(\det(Q_x + \epsilon I))
\]

B

s. t. \( \text{trace}(\alpha^{(m)} \alpha^{(m)*} Q_x) = c^{(m)} \forall m = 1, \ldots, M \)

\( Q_x \geq 0 \)

- Eigenvectors of \( Q_{\hat{x}_n} := \hat{x}_n \hat{x}_n^* \): \( \hat{x}_n e^{i\theta} \forall \theta \in [-\pi, \pi] \)

Finite dimensional signal recovery only up to constant phase

---

Sampling in the complex plane

Motivation

Main results

Corollary

Discussion

Setup

Parameters

Reconstruction

Finite Dimensional Phase Retrieval

Intensity measurement

Sampling with rate $1/\beta$

$\hat{x}_n e^{i\theta_n}$
Sampling in the complex plane

\[ |\hat{\chi}^{(1)}(\omega)| : \quad n \beta \quad \cdots \quad (n+1) \beta \quad \cdots \quad (n+2) \beta \quad \text{Frequency (} \omega \text{)} \]

\[ |\hat{\chi}^{(M)}(\omega)| : \quad \cdots \quad \text{Frequency (} \omega \text{)} \]

\[ \{ |\langle \alpha^{(m)}, \hat{\chi}_n \rangle|^2 \}_{m=1,\ldots,M} \quad \downarrow \quad \{ |\langle \alpha^{(m)}, \hat{\chi}_{n+1} \rangle|^2 \}_{m=1,\ldots,M} \]

Finite Dimensional Phase Retrieval

\[ \{ \hat{\chi}(n \beta + \tilde{\lambda}_k) e^{i \theta_n} \}_{k=1,\ldots,K} \quad \downarrow \quad \{ \hat{\chi}((n+1) \beta + \tilde{\lambda}_k) e^{i \theta_{n+1}} \}_{k=1,\ldots,K} \]

\[ \hat{\chi}(z) : \quad \text{Im}(z) \quad \text{Re}(z) \]

...
### Phase Retrieval – Approach overview

| Problem formulation | Given $c_n^{(m)} = |\phi_n^{(m)}(\hat{x})|^2$ for $m = 1, \ldots, M$, $n \in \mathbb{Z}$ | Recover $\hat{x} \in \mathcal{PW}_{T/2}$ |

Goal: Find measurement scheme and thus $\phi_n^{(m)}(\cdot)$ leading to the subproblems:

1. Finite dimensional phase retrieval $\rightarrow \hat{x}_n \ \forall n \in \mathbb{Z}$

2. Interpolation from samples $\{\hat{x}_n\}_{n\in\mathbb{Z}} \rightarrow \hat{x}$
Phase propagation

Motivation  Main results  Corollary  Discussion  Setup  Parameters  Reconstruction

Choice of $\tilde{\lambda}_k$:

- Consecutive finite blocks should have at least one overlap.
- $\hat{x} \neq 0$ at these overlaps!
Sampling in the complex plane

Relabelling all sampling points \( \Lambda = \{ \lambda_n \}_{n \in \mathbb{Z}} \)

Recall:

2. Interpolation from samples \( \{ \hat{x}_n e^{i\theta_n} \}_{n \in \mathbb{Z}} \rightarrow \hat{x} \)

\[
\begin{align*}
\{ \hat{x}_n e^{i\theta_n} \}_{n \in \mathbb{Z}} & \rightarrow \{ \hat{x}(\lambda_n) e^{i\theta_0} \}_{n \in \mathbb{Z}} \\
& \rightarrow \hat{x} e^{i\theta_0}
\end{align*}
\]

Choice of \( \lambda_n \)?
Choice of $\lambda_n$ – Complete Interpolating Sequences

Interpolation condition: $\hat{x}_a(\lambda_n) = \hat{x}(\lambda_n)$ $\forall n \in \mathbb{Z}$
Solved by: $\hat{x}_a(z) = \sum_{n \in \mathbb{Z}} \hat{x}(\lambda_n) \hat{\psi}_n(z)$

Choice of $\lambda_n$ for perfect reconstruction such that
$\hat{x}_a(z) = \hat{x}(z)$ $\forall z \in \mathbb{C}$
$\hat{x}_a(\lambda_n) = \hat{x}(\lambda_n)$ $\forall n \in \mathbb{Z}$ solved uniquely
$\{\lambda_n\}_{n \in \mathbb{Z}}$ complete interpolating for $\mathcal{PW}_{T/2}$

Nice result for Paley Wiener spaces
Choice of $\lambda_n$ – Zeros of sine-type functions

For $\hat{x} \in \mathcal{PW}_{T/2}$

\[
\{\lambda_n\}_{n \in \mathbb{Z}} \text{ is a complete interpolating sequence}
\]

\[
\{e^{i\lambda_t}\}_{n \in \mathbb{Z}} \text{ is a Riesz basis for } L^2(\mathbb{T})
\]

e.g. zeros of sine-type functions $S(z)$ of type $\geq T/2$

of the form $S(z) = P.V. \prod_{n \in \mathbb{Z}} (1 - \frac{z}{\lambda_n})$

\[
\hat{x}(z) = \sum_{n \in \mathbb{Z}} \hat{x}(\lambda_n) \frac{S(z)}{S'(\lambda_n)(z - \lambda_n)} = \sum_{n \in \mathbb{Z}} \hat{x}(\lambda_n) \prod_{m \neq n} \frac{z - \lambda_m}{\lambda_n - \lambda_m}
\]

Example (Sine-type functions)

For $\mathbb{T} = [-\pi, \pi]$, $\sin(z)$ is of sine-type $\pi$ $\rightarrow$ $\{\lambda_n := n\}_{n \in \mathbb{Z}}$

Shannon series: $\hat{x}(z) = \sum_{n \in \mathbb{Z}} \hat{x}(n) \frac{\sin(\pi(z - n))}{\pi(z - n)}$

Main Theorem

Given the measurement setup and $p^{(m)}$ as above. Then $\hat{x} \in \mathcal{PW}_{T/2}$ can be perfectly recovered from $c^{(m)}_n = |\langle \alpha^{(m)}, \hat{x}_n \rangle|^2$ for $m = 1, \ldots, M$, $n \in \mathbb{Z}$ whenever

1. $\{\alpha^{(m)}\}$ constitutes a 2-uniform M/K tight frame with $M = K^2$ \[ \hat{x}_n \mapsto c^{(m)}_n \text{ injective} \]

2a. $\bar{\lambda}_k$ s.t. consecutive blocks have at least one overlap with $\hat{x} \neq 0$ \[ \{\hat{x}_n e^{i\theta_n}\}_{n \in \mathbb{Z}} \mapsto \{\hat{x}(\lambda_n)\}_{n \in \mathbb{Z}} \text{ is defined} \]

2b. $\{\lambda_n\}_{n \in \mathbb{Z}}$ is a complete interpolating sequence \[ \{\hat{x}(\lambda_n)\}_{n \in \mathbb{Z}} \mapsto \hat{x} \text{ unique} \]
How can we ensure that $\hat{x}(\lambda) \neq 0$ at the overlapping sampling points?

**Corollary**

Let the maximal energy of $x$ be known $\|x\|_{L^2(\mathbb{R})} \leq W_0$

Consider the following function as our signal in the Fourier domain with $T' \geq T$:

$$\hat{v}(z) = D \cos\left(\frac{T'}{2} z\right) - \hat{x}(z)$$

- when $D$ is large enough, the zeros of $\hat{v}$ are concentrated in a strip $|\text{Im}(z)| < H$

- then, one can find $\{\lambda_n\}$ for perfect reconstruction of $x$ from $c_n^{(m)} = |\langle \alpha^{(m)}, \hat{v}_n \rangle|^2$ up to a constant phase

Example construction of feasible $\lambda_n$
Avoiding samples at signal zeroes

Theorem (Levin)

By shifting the imaginary parts of the zeros of a sine-type function, the corresponding function

\[ S(z) = P.V. \prod_{n \in \mathbb{Z}} \left(1 - \frac{z}{\lambda_n}\right) \]

remains to be a sine-type function, i.e. the resulting zeros are still a complete interpolating sequence.

Perfect signal reconstruction from magnitude measurements for $\mathcal{L}^2(\mathbb{T}) \leftrightarrow \mathcal{P}\mathcal{W}_{T/2}$ by using special structure of the modulators

$$\rho^{(m)}(t) := \sum_{k=1}^{K} \alpha_k^{(m)} e^{i\tilde{\lambda}_k t}, \ m = 1, \ldots M , \tilde{\lambda}_k \in \mathbb{C}$$

Overlap non-zero condition unnecessary when maximal energy $\|x\|_{\mathcal{L}^2(\mathbb{T})} \leq W_0$ of the signal is given

For $K = 2$ and one overlap, we obtain the minimal overall sampling rate $R = 4R_{Ny}$ with the nyquist rate $R_{Ny} = \frac{T}{2\pi}$
Outlook

Open questions:

- Extension to bigger signal spaces, e.g. to Bernstein spaces with \( \sup_{\omega \in \mathbb{R}} |\hat{x}(\omega)| < \infty \)

- Behavior under noise disturbance?

- Extension to the 2-dimensional case?

---

Thank you!
Choice of $\lambda_n$ – Zeros of sine-type functions

For $x \in \mathcal{P}_{T/2}$

**Theorem (Complete Interpolating Sequence)**

\[ \{\lambda_n\}_{n \in \mathbb{Z}} \text{ is a complete interpolating sequence} \]

\[ \{e^{i\lambda_nt}\}_{n \in \mathbb{Z}} \text{ is a Riesz basis for } L^2(\mathbb{T}) \]

One example: Zeros of sine-type functions of type $\geq T/2$

**Definition (Sine-type functions)**

An entire function of exponential type $T/2$ \( S(z) = \text{P.V.} \prod_{n \in \mathbb{Z}}(1 - \frac{z}{\lambda_n}) \) is called sine-type of type $T/2$ if it has simple and separated zeros $\lambda_n$, for which there exist $A, B, H$ s.t.

\[ Ae^{\frac{T}{2}|\eta|} \leq |S(\xi + i\eta)| \leq Be^{\frac{T}{2}|\eta|}, \text{ for } |\eta| > H. \]