Phase Retrieval via Structured Modulations In Paley Wiener Spaces

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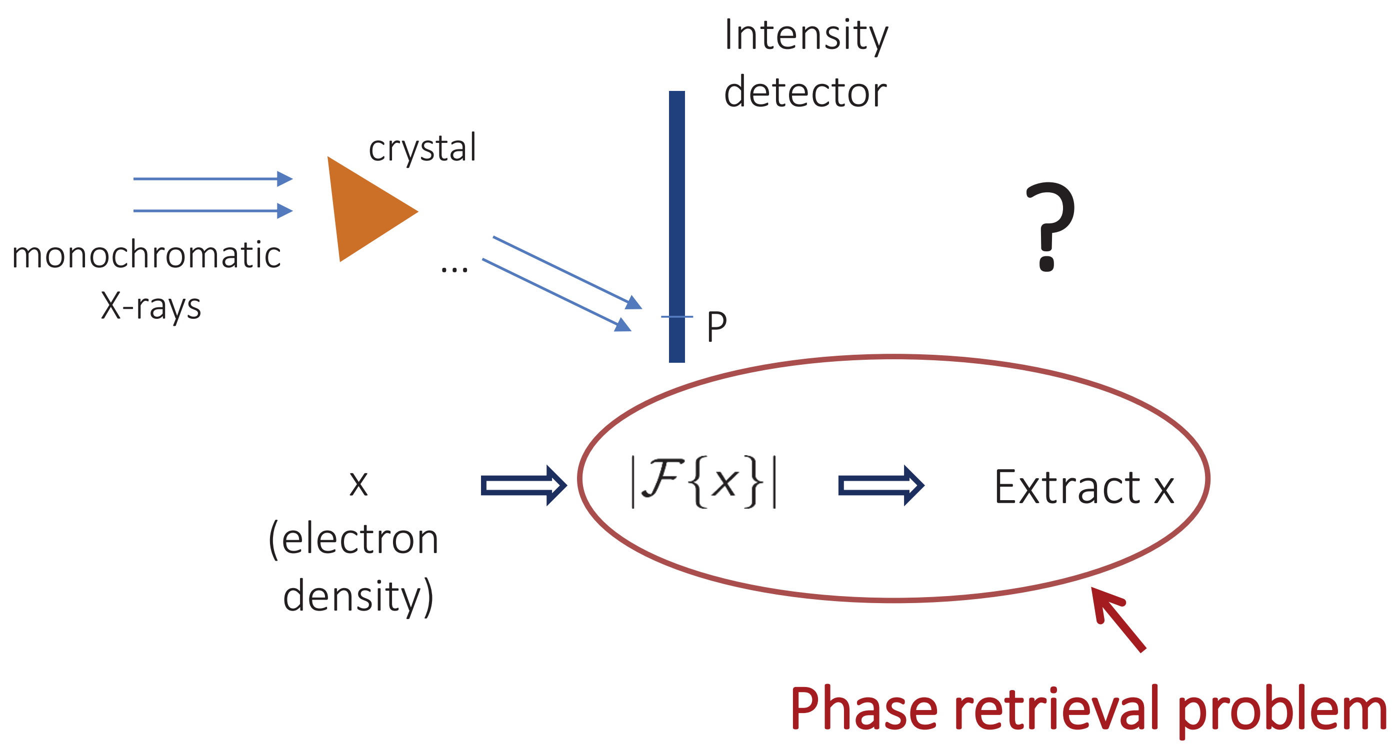
SampTA 2013, Jacobs University Bremen
July 4th, 2013

Overview

- Phase Retrieval
  - Applications
  - Finite dimensional
- Main result – Paley-Wiener spaces
  - Measurement setup
  - Choice of measurement parameters
- Corollary
- Discussion and outlook

Phase Retrieval - Motivation

X-Ray crystallography



Phase Retrieval – K dimensions

Signal processing approach: K-dimensional

Problem formulation	Given $c_m = |\langle a_m, x\rangle|^2$ for $m = 1, \ldots, M$ Recover $x \in \mathbb{C}^K$

⟹ Determine a_m and M

When is $x \mapsto \langle a_m, x\rangle$ injective?

- Choose $\{a_m\}$ as basis
- $M = K$ (e.g. DFT)

When is $x \mapsto |\langle a_m, x\rangle|^2$ injective?

[Balan et al. 2009, Balan 2006]

- Generic frame: $M \geq 4K\text{-}2$ (or 4K-4?)
- Choose $\{a_m\}$ as MUB or as a 2-uniform M/K tight frame with $M = K^2$

R. Balan, B. G. Bodmann, P. G. Casazza, D. Edidin, „Painless reconstruction from magnitudes of frame coefficients “, J. Fourier Anal. Appl., vol. 15 (Aug. 2009), no. 4, 488-501.
R. Balan, et al. ,"On signal reconstruction without phase." *Applied and Computational Harmonic Analysis* 20.3 (2006): 345-356.

Phase Retrieval - Motivation

K dimensions ⟹ ∞ dimensions ?

Phase Retrieval – ∞-dimensional signal space

Time (spatially) limited signals $x \in \mathcal{L}^2(\mathbb{T})$ with $\mathbb{T} = [-\frac{T}{2}, \frac{T}{2}]$

Define „Fourier Transform“: $x \mapsto \hat{x}(z) := \int_{\mathbb{T}} x(t)\, \mathrm{e}^{itz}\, \mathrm{d}t\,, \forall z \in \mathbb{C}$

⇒ $x \in \mathcal{L}^2(\mathbb{T}) \Leftrightarrow \hat{x}(z) \in \mathcal{PW}_{T/2}$

- $\hat{x}(z)$ is an entire function with $|\hat{x}(z)| \leq C\mathrm{e}^{\frac{T}{2}|z|}$
- $\int_{\mathbb{R}} |\hat{x}(\omega)|^2 \mathrm{d}\omega < \infty$

⇒ Isomorphic with inverse:

$$x(t) = \frac{1}{T} \int_{\mathbb{R}} \hat{x}(\omega) \mathrm{e}^{it\omega}\, \mathrm{d}\omega \;\; \forall t \in \mathbb{T}$$

Recovery of x(t) ↔ Reconstruction of $\hat{x}(z)$

Hörmander, Lars. *The Analysis of Linear Partial Differential Operators: Vol.: 1.: Distribution Theory and Fourier Analysis*. Springer-Verlag, 1983.

Phase Retrieval – Approach overview

$x \mapsto |\langle a_m, x\rangle|^2 \quad \Rightarrow \quad \hat{x} \mapsto |\phi_n^{(m)}(\hat{x})|^2$

with linear measurement functional $\phi_n^{(m)}(\cdot) : \mathcal{PW}_{T/2} \to \mathbb{C}$
determined by specific measurement setup

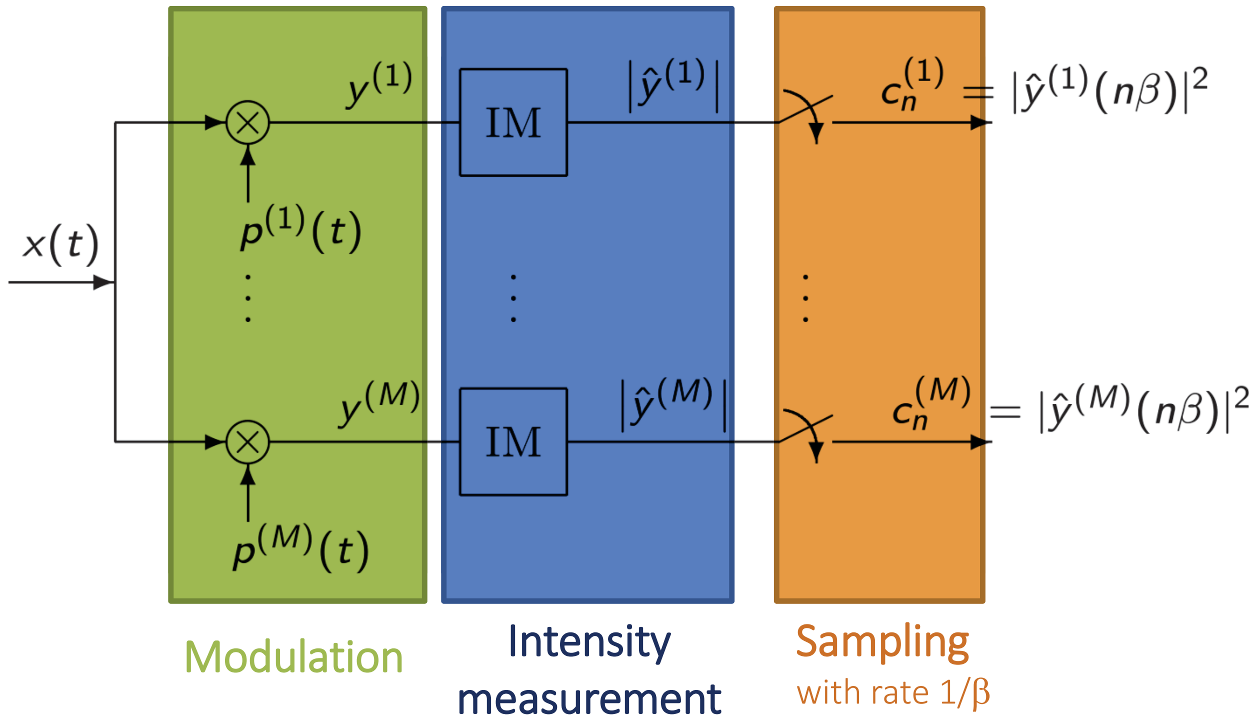
Problem formulation	Given $c_n^{(m)} = |\phi_n^{(m)}(\hat{x})|^2$ for $m = 1, \dots, M,\ n \in \mathbb{Z}$ Recover $\hat{x} \in \mathcal{PW}_{T/2}$

Goal: Find measurement scheme and thus $\phi_n^{(m)}(\cdot)$ leading to the subproblems:

① Finite dimensional phase retrieval

② Interpolation from samples

Our approach – Measurement setup



$$p^{(m)}(t) := \sum_{k=1}^{K} \overline{\alpha_k^{(m)}} e^{i\tilde{\lambda}_k t}, \; m = 1, \dots M, \; \tilde{\lambda}_k \in \mathbb{C}$$

$$\alpha^{(m)} := \begin{pmatrix} \alpha_1^{(m)} \\ \vdots \\ \alpha_K^{(m)} \end{pmatrix} \quad \hat{x}_n := \begin{pmatrix} \hat{x}(n\beta + \tilde{\lambda}_1) \\ \vdots \\ \hat{x}(n\beta + \tilde{\lambda}_K) \end{pmatrix}$$

$$c_n^{(m)} := \left| \sum_{k=1}^{K} \overline{\alpha_k^{(m)}} \hat{x}(n\beta + \tilde{\lambda}_k) \right|^2 = |\underbrace{\langle \alpha^{(m)}, \hat{x}_n \rangle_{\mathbb{C}^K}}_{\phi_n^{(m)}(\hat{x})}|^2$$

For fix n as in K-dimensional phase retrieval!

Choice of modulator coefficients $\alpha^{(m)}$

① For each n solve

$$\{|\langle\alpha^{(m)},\hat{x}_n\rangle|^2\}_{m=1,\dots,M} \to \hat{x}_n$$

⟹ Choice of vectors $\alpha^{(m)}$?

How big is M?

- Sufficient condition: $\{\alpha^{(m)}\}$ is 2-uniform M/K-tight, M = K^2
- Instead of $\hat{x}_n$ obtain $Q_{\hat{x}_n} := \hat{x}_n\hat{x}_n^*$

A [Balan et al., 2009]

$$Q_{\hat{x}_n} = \frac{(K+1)}{K}\sum_{m=1}^{M} c_n^{(m)} Q_{\alpha^{(m)}} - \frac{1}{K}\sum_{m=1}^{M} c_n^{(m)} I$$

B [Candes et al., 2013]

$$\min_{Q_x} \log(\det(Q_x + \epsilon I))$$

$$\text{s.t. } \operatorname{trace}(\alpha^{(m)}\alpha^{(m)*}Q_x) = c^{(m)} \ \forall m = 1, \dots, M$$

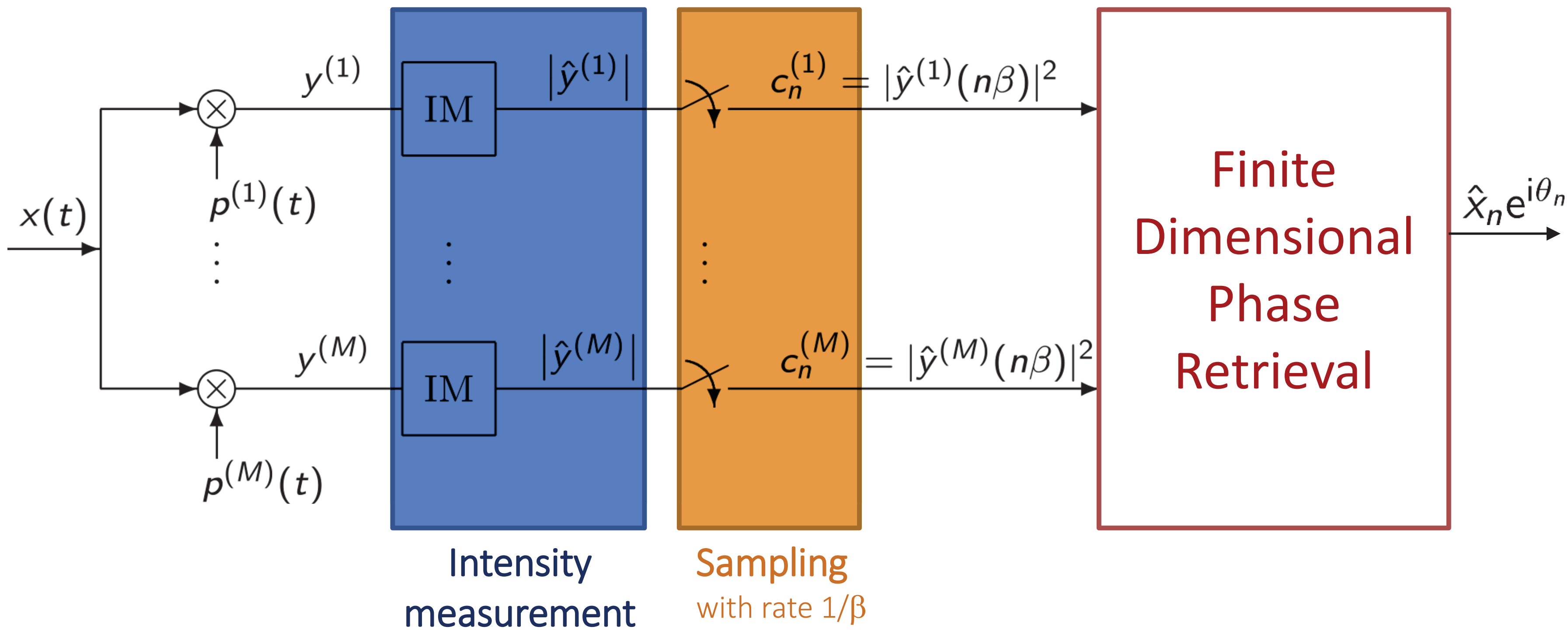
$$Q_x \geq 0$$

- Eigenvectors of $Q_{\hat{x}_n} := \hat{x}_n\hat{x}_n^*$: $\hat{x}_n e^{i\theta}\ \forall\theta \in [-\pi,\pi]$

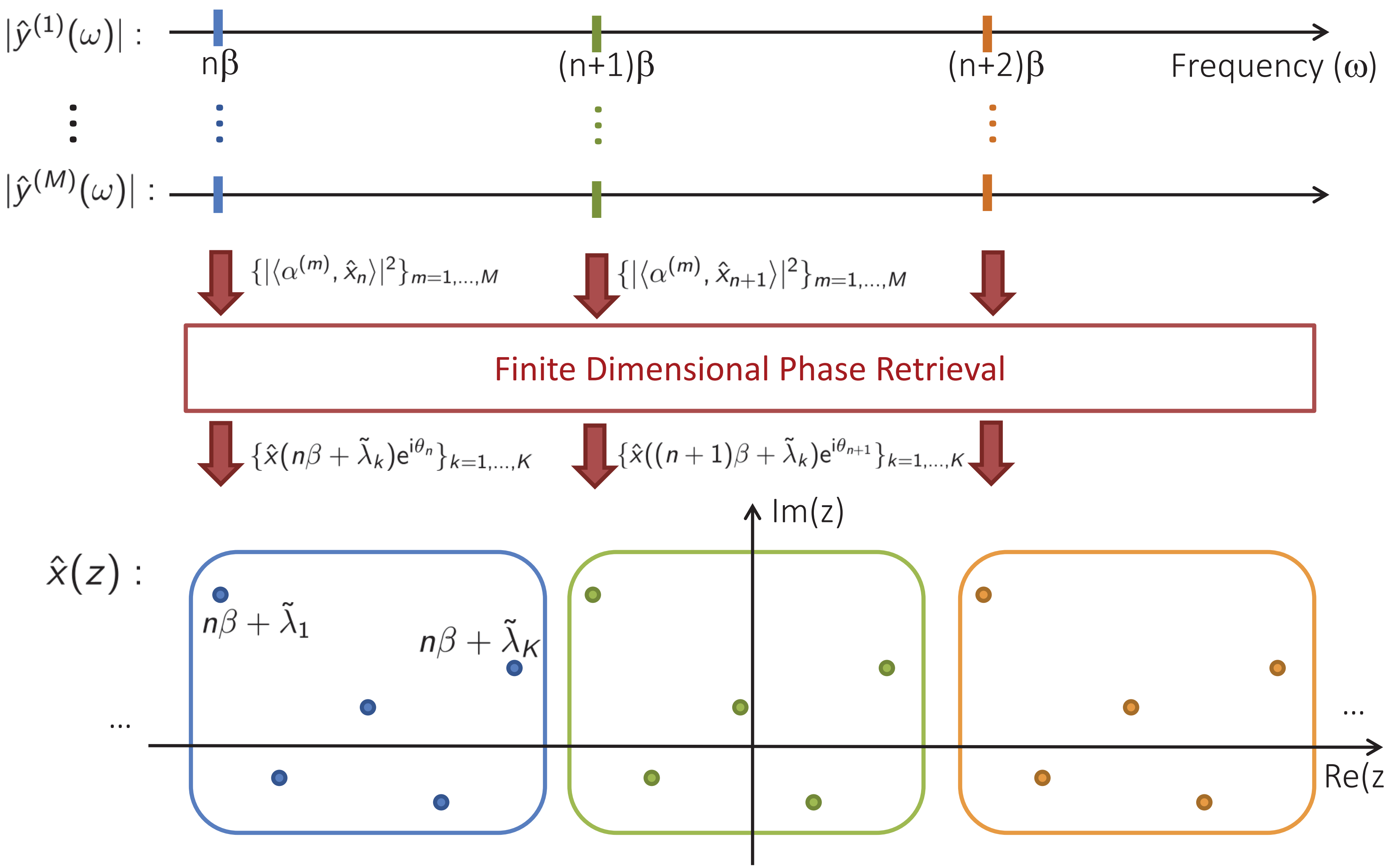
⟹ **Finite dimensional signal recovery only up to constant phase**

R. Balan et al., „Painless reconstruction from magnitudes of frame coefficients “, J. Fourier Anal. Appl., vol. 15 (Aug. 2009), no. 4, 488-501.
E. J. Candès, Y. C. Eldar, T. Strohmer, V. Voroninski, „Phase Retrieval via Matrix Completion “, SIAM J. Imaging Sci. vol. 6 (2013), no. 1, 199-255.

Sampling in the complex plane



Sampling in the complex plane



Phase Retrieval – Approach overview

Problem formulation	Given $c_n^{(m)} = \|\phi_n^{(m)}(\hat{x})\|^2$ for $m = 1, \dots, M, \;\; n \in \mathbb{Z}$ Recover $\hat{x} \in \mathcal{PW}_{T/2}$

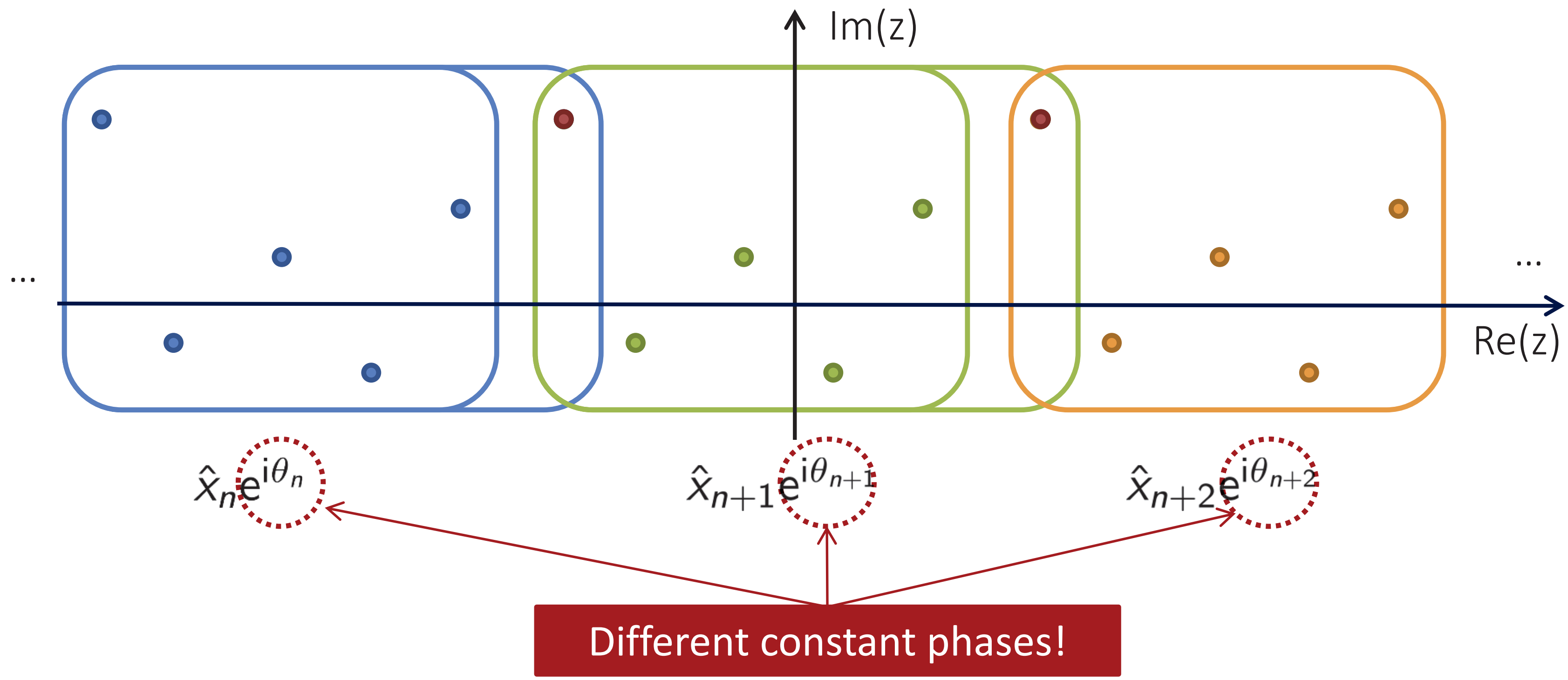
Goal: Find measurement scheme and thus $\phi_n^{(m)}(\cdot)$ ✓

leading to the subproblems:

① Finite dimensional phase retrieval $\rightarrow \hat{x}_n \; \forall n \in \mathbb{Z}$ ✓

② Interpolation from samples $\{\hat{x}_n\}_{n\in\mathbb{Z}} \dashrightarrow \hat{x}$

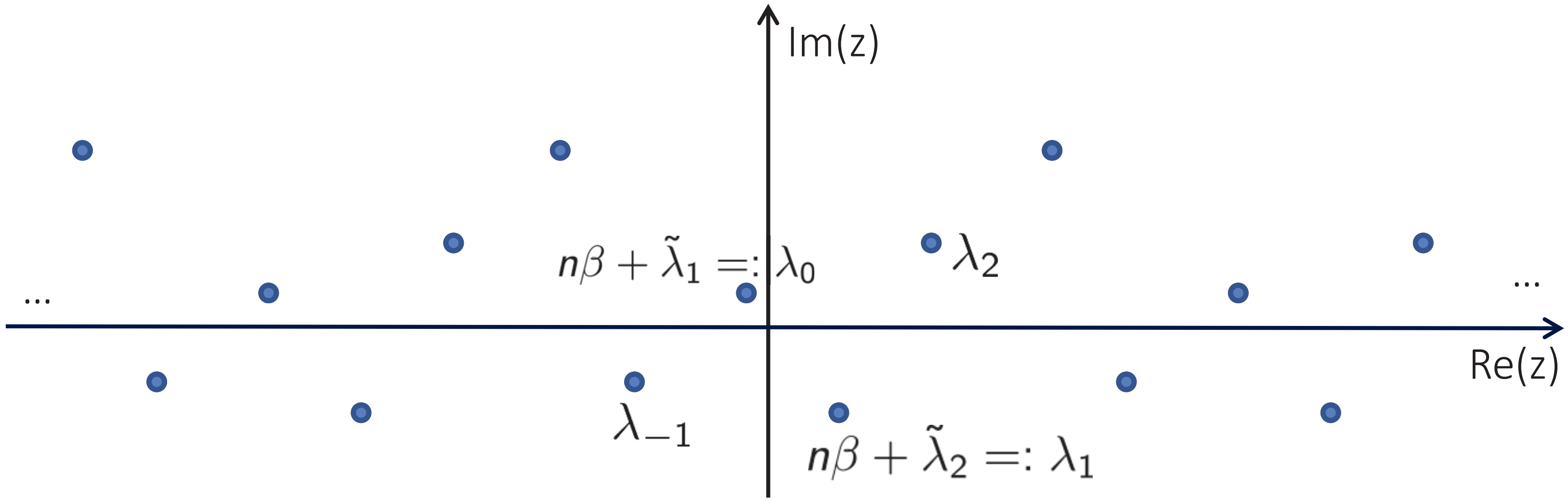
Phase propagation



ⓐ Choice of $\tilde{\lambda}_k$:

- Consecutive finite blocks should have at least one overlap
- $\hat{x} \overset{!}{\neq} 0$ at these overlaps!

Sampling in the complex plane



Relabelling all sampling points $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$

Recall: ② Interpolation from samples $\{\hat{x}_n e^{i\theta_n}\}_{n\in\mathbb{Z}} \dashrightarrow \hat{x}$

$$\{\hat{x}_n e^{i\theta_n}\}_{n\in\mathbb{Z}} \xrightarrow{ⓐ ✓} \{\hat{x}(\lambda_n) e^{i\theta_0}\}_{n\in\mathbb{Z}} \xrightarrow{ⓑ} \hat{x} e^{i\theta_0}$$

⟹ Choice of λ_n?

Choice of λ_n – Complete Interpolating Sequences

② $\{\hat{x}_n e^{i\theta_n}\}_{n\in\mathbb{Z}} \xrightarrow{ⓐ✓} \{\hat{x}(\lambda_n)\}_{n\in\mathbb{Z}} \xrightarrow{ⓑ} \hat{x}_a$

⇒ Interpolation condition: $\hat{x}_a(\lambda_n) = \hat{x}(\lambda_n)\ \forall n \in \mathbb{Z}$

Solved by: $\hat{x}_a(z) = \sum_{n\in\mathbb{Z}} \hat{x}(\lambda_n)\hat{\psi}_n(z)$

⇒ Choice of λ_n for perfect reconstruction such that

$\hat{x}_a(z) = \hat{x}(z)\ \forall z \in \mathbb{C}$

⇔ $\hat{x}_a(\lambda_n) = \hat{x}(\lambda_n)\ \forall n \in \mathbb{Z}$ solved uniquely

⇔: $\{\lambda_n\}_{n\in\mathbb{Z}}$ complete interpolating for $\mathcal{PW}_{T/2}$

Nice result for Paley Wiener spaces

Choice of λ_n – Zeros of sine-type functions

For $\hat{x} \in \mathcal{PW}_{T/2}$

Theorem (Complete Interpolating Sequence)

$\{\lambda_n\}_{n\in\mathbb{Z}}$ is a complete interpolating sequence

$\Longleftrightarrow \{\mathrm{e}^{\mathrm{i}\lambda_n t}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{L}^2(\mathbb{T})$

e.g. zeros of sine-type functions ***S(z)*** of type ≥T/2

of the form $S(z) = P.V. \prod_{n\in\mathbb{Z}}(1 - \frac{z}{\lambda_n})$

$$\Rightarrow \quad \hat{x}(z) = \sum_{n\in\mathbb{Z}} \hat{x}(\lambda_n) \frac{S(z)}{S'(\lambda_n)(z-\lambda_n)} = \sum_{n\in\mathbb{Z}} \hat{x}(\lambda_n) \prod_{m\neq n} \frac{z-\lambda_m}{\lambda_n - \lambda_m}$$

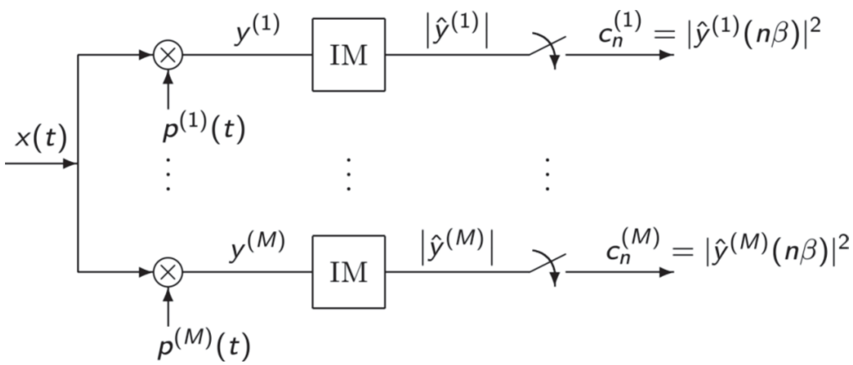
Example (Sine-type functions)

For $\mathbb{T} = [-\pi, \pi]$, $\sin(z)$ is of sine-type π $\Rightarrow$ $\{\lambda_n := n\}_{n\in\mathbb{Z}}$

Shannon series: $\hat{x}(z) = \sum_{n\in\mathbb{Z}} \hat{x}(n) \frac{\sin(\pi(z-n))}{\pi(z-n)}$

B. Y. Levin, „Lectures on entire functions“, American Mathematical Society, Providence, RI, 1997.

Main Theorem



$$p^{(m)}(t) := \sum_{k=1}^{K} \overline{\alpha_k^{(m)}} e^{i\tilde{\lambda}_k t}, \; m = 1, \dots M, \; \tilde{\lambda}_k \in \mathbb{C}$$

$$\alpha^{(m)} := \begin{pmatrix} \alpha_1^{(m)} \\ \vdots \\ \alpha_K^{(m)} \end{pmatrix} \qquad \hat{x}_n := \begin{pmatrix} \hat{x}(n\beta + \tilde{\lambda}_1) \\ \vdots \\ \hat{x}(n\beta + \tilde{\lambda}_K) \end{pmatrix}$$

Main Theorem

Given the measurement setup and $p^{(m)}$ as above.
Then $\hat{x} \in \mathcal{PW}_{T/2}$ can be perfectly recovered from $c_n^{(m)} = |\langle \alpha^{(m)}, \hat{x}_n \rangle|^2$ for $m = 1, \dots, M$, $n \in \mathbb{Z}$ whenever

① $\{\alpha^{(m)}\}$ constitutes a 2-uniform M/K tight frame with M = K^2 ‑‑‑→ $\hat{x}_n \mapsto c_n^{(m)}$ injective

②a $\tilde{\lambda}_k$ s.t. consecutive blocks have at least one overlap with $\hat{x} \neq 0$ ‑‑‑→ $\{\hat{x}_n e^{i\theta_n}\}_{n\in\mathbb{Z}} \mapsto \{\hat{x}(\lambda_n)\}_{n\in\mathbb{Z}}$ is defined

②b $\{\lambda_n\}_{n\in\mathbb{Z}}$ is a complete interpolating sequence ‑‑‑→ $\{\hat{x}(\lambda_n)\}_{n\in\mathbb{Z}} \mapsto \hat{x}$ unique

Corollary – Signals with known maximal energy

How can we ensure that $\hat{x}(\lambda) \neq 0$ at the overlapping sampling points?

Corollary

Let the maximal energy of x be known $\|x\|_{\mathcal{L}^2(\mathbb{T})} \leq W_0$

Consider the following function as our signal in the Fourier domain with $T' \geq T$

$$\hat{v}(z) = D\cos\left(\frac{T'}{2}z\right) - \hat{x}(z)$$

⟹ when D is large enough, the zeros of $\hat{v}$ are concentrated in a strip $|\operatorname{Im}(z)| < H$

⟹ then, one can find $\{\lambda_n\}$ for perfect reconstruction of x from $c_n^{(m)} = |\langle \alpha^{(m)}, \hat{v}_n \rangle|^2$ up to a constant phase

Example construction of feasible λ_n ⟹

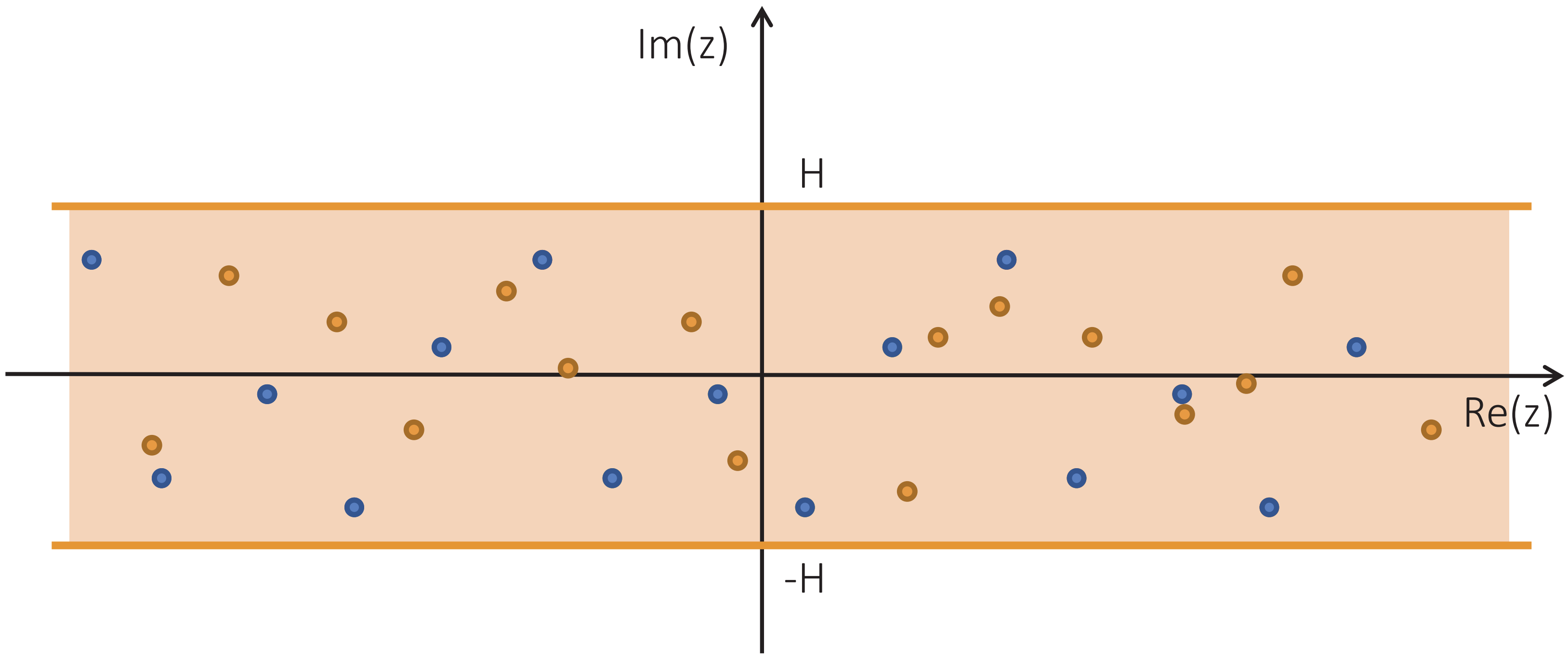
Avoiding samples at signal zeroes

Theorem (Levin)

By shifting the imaginary parts of the zeros of a sine-type function, the corresponding function

$$S(z) = P.V. \prod_{n \in \mathbb{Z}} \left(1 - \frac{z}{\lambda_n}\right)$$

remains to be a sine- type function, i.e. the resulting zeros are still a complete interpolating sequence.



B. Levin and I. Ostrovskii, "Small perturbations of the set of roots of sine-type functions," Izv. Akad. Nauk SSSR Ser. Mat, vol. 43, pp. 87–110, 1979.

Summary and discussion

- Perfect signal reconstruction from magnitude measurements for $x \in \mathcal{L}^2(\mathbb{T}) \leftrightarrow \hat{x} \in \mathcal{PW}_{T/2}$ by using special structure of the modulators
  $p^{(m)}(t) := \sum_{k=1}^{K} \overline{\alpha_k^{(m)}} \mathrm{e}^{\mathrm{i}\tilde{\lambda}_k t}$, $m = 1, \dots M$, $\tilde{\lambda}_k \in \mathbb{C}$

- Overlap non-zero condition unnecsesary when maximal energy $\|x\|_{\mathcal{L}^2(\mathbb{T})} \leq W_0$ of the signal is given

- For K = 2 and one overlap, we obtain the minimal overall sampling rate $R = 4R_{\mathrm{Ny}}$ with the nyquist rate $R_{\mathrm{Ny}} = \frac{T}{2\pi}$

Outlook

Open questions:

- Extension to bigger signal spaces, e.g. to Bernstein spaces with $\sup_{\omega \in \mathbb{R}} |\hat{x}(\omega)| < \infty$ [Pohl, Yang, Boche, 2013]
- Behavior under noise disturbance?
- Extension to the 2-dimensional case?

V. Pohl, F. Yang, H. Boche, „Phaseless Signal Recovery in Infinite Dimensional Spaces using Stuctured Modulations ", preprint: arXiv:1305.2789 (May 2013).

Thank you!

Choice of λ_n – Zeros of sine-type functions

For $\hat{x} \in \mathcal{PW}_{T/2}$

Theorem (Complete Interpolating Sequence)

$\{\lambda_n\}_{n\in\mathbb{Z}}$ is a complete interpolating sequence

$\Longleftrightarrow \{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{L}^2(\mathbb{T})$

One example: Zeros of sine-type functions of type ≥T/2

Definition (Sine-type functions)

An entire function of exponential type T/2 $S(z) = P.V. \prod_{n\in\mathbb{Z}}(1 - \frac{z}{\lambda_n})$ is called sine-type of type T/2 if it has simple and separated zeros λ_n, for which there exist A, B, H s.t.

$$A\,e^{\frac{T}{2}|\eta|} \leq |S(\xi + i\eta)| \leq B\,e^{\frac{T}{2}|\eta|}\,, \quad \text{for } |\eta| > H\,.$$

B. Y. Levin, „Lectures on entire functions“, American Mathematical Society, Providence, RI, 1997.