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Signal Reconstruction from Magnitude Measurements in Infinite Dimensional Spaces

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Abstract

This thesis provides a mathematical framework for the recovery of continuous signals in infinite dimensional spaces from the magnitude of their frequency samples. This problem, known as phase retrieval, is an important problem in many imaging systems which has recently shifted into the focus in its reduced finite dimensional version. By proposing a special sampling scheme involving a combination of oversampling and structured modulations, it is possible to use the results for the infinite dimensional case. After the introduction of spaces for timelimited signals, unique interpolation on these spaces is discussed upon which the reconstruction procedure can be established. This method allows for almost every signal with finite support to be reconstructed from its magnitude samples in the frequency domain up to a unimodular constant. Sufficient conditions on the signal and the sampling system are given such that signal recovery is possible, and it is shown that an average sampling rate of four times the Nyquist rate is enough to reconstruct almost every time limited signal.

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Contents

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Introduction

1.1 Phase Retrieval Problem

Signal reconstruction from magnitude measurements is a necessary step in many different applications of engineering and science. It is well-known as the *phase retrieval problem* and originates from the fact that detectors can often only record the squared modulus of an electromagnetic wave while its phase is not acquired. One of its most important applications is in X-ray crystallography, widely used in material and biological science [27, 28].

it also appears in numerous other areas of imaging science including electron microscopy, astronomical imaging, diffraction imaging, X-ray tomography, to mention only a few. Furthermore phase retrieval plays an important role in other fields such as speech processing [31], radar [20], signal theory [17], quantum communication [15].

In the case of X-ray crystallography, a crystal is illuminated by a monochromatic X-ray source upon which each electron absorbs and reemits the light as a point source. One can then see that the light field on a distant plane behind the crystal is the Fourier Transform of crystal's electron density. However, the detectors in X-ray imaging are usually based on photon counting or measured energy, so that the phase of the light field is lost. As the aim is to obtain the original electron density, the Inverse Fourier Transform generally requires the phase to depict the desired structure of the crystal. Therefore we speak of phaseless recovery.

The underlying problem is the fact that in general, magnitude and phase of a signal are independent so that the information is irreversibly lost. Therefore signal recovery from magnitude measurements is only possible if it is combined with additional information about the signal. If, for example, the signal is known to be causal or bandlimited, then the logarithm of the magnitude and its phase are related by the Hilbert transform so that the intensity information can be directly used to recover the phase [6, 34]. Similarly, the signal is completely determined by its magnitude if the z-transform satisfies certain sufficient conditions [17],.

When no or only little a priori knowledge is available on the original signal, one can still successfully perform reconstruction from magnitude information by taking several measurements of the same object under slightly different conditions. To this end various methods were proposed, such as the distorted-object approach by which the Fresnel diffraction pattern is measured at different distances [41], the usage of aperture-plane modulation [13, 44], or the recording of several fractional Fourier transforms [20]. Having obtained the different measurements, signal recovery is mainly performed by iterative alternating projection algorithms [14]. Although these algorithms are usually easy to implement, their convergence strongly depends on specific signal constraints (see, e.g., [4,26]). Moreover there seems to be no systematic approach to design the different measurements such that iterative convergence to the correct solution is guaranteed.

Recently, analytic investigations on the phase retrieval problem have revealed sufficient conditions on the number and types of measurements such that a unique solution exists. In [2, 3, 5] it was found that for an N-dimensional space slightly less than $4N$ suitably chosen amplitude measurements are sufficient. However, no corresponding reconstruction algorithm was given which achieves this bound. The closest result was presented in [1], where the authors proposed a method which guarantees signal recovery from amplitude measurements requiring N^2 measurements. This fundamentally limits its practicability to low dimensional spaces. Ideas of sparse signal representation and convex optimization were applied in [7, 25] to allow for lower computational complexity.

Note that all of the above approaches address finite dimensional signals. A natural follow-up question is whether similar results can be obtained for continuous signals in infinite dimensional spaces. In [38] it was shown that real valued bandlimited signals are completely determined simply by their magnitude samples taken at twice the Nyquist rate. However this result cannot be extended in a straight forward manner for complex valued signals, since results for finite dimensional spaces indicate that oversampling alone may not be sufficient. In [1, 2, 7], the particular choice of measurement vectors was the key to enabling signal recovery.

During the course of this masters thesis, the first attempt on phaseless recovery of complex valued time limited $\mathcal{L}^2$-signals was presented in [42]. In this report, this signal space will be treated as a special case of the more general result for larger signal spaces $\mathcal{L}^p$ for $1 \leq p \leq \infty$ with compact supports, which has also appeared in the preprint [30].

The main idea is to choose the measurement setup such that we can use well-known results for finite dimensional phase retrieval and classical interpolation in Bernstein spaces. Inspired by the experimental approaches in optics [7, 13, 41, 44], we choose the method of structured modulations which turns out to be a legitimate choice to achieve this goal. We then derive conditions on the modulators so that generically (i.e. up to a meager set) every signal can be reconstructed up to a unimodular constant. In the proof an algorithm is provided which only requires samples taken at a rate arbitrarily close to four times the Nyquist rate.

1.2 Thesis organization

In the first two chapters, the necessary mathematical tools and background will be introduced. In Chapter 2, distributions are introduced which are a broader topological space than the Lebesgue functions and enable us to handle the Bernstein spaces $\mathcal{B}_\sigma^p$ which, for $p > 2$, have no well-defined Fourier Inverse Transform in the sense of a function. Entire functions theory, which is discussed in Chapter 3, then provides us with interpolation results for $\mathcal{B}_\sigma^p$ and is the main building block to shift phase retrieval from finite to infinite dimensions. Then finally in Chapter 4 the status quo of finite phase retrieval is briefly recaptured before we use the background gained in the previous two chapters to establish our main reconstruction results. The thesis ends with a brief discussion on robustness and an outlook to potential future work.

2

Distribution Theory

The ultimate goal in this chapter is to gain insights on the Fourier Transforms on $\mathcal{L}^p$ spaces, in one of which a signal usually lives in. For this purpose we need generalized functions in the space of distributions $\mathcal{D}'$ which are defined as functionals on test functions in $\mathcal{D}$. They cannot necessarily be associated with a classical function which have a given value for each point. The Schwartz space $\mathcal{S}$ and its dual space of tempered distributions $\mathcal{S}'$ are then introduced because the Fourier Transform is bijective and continuous on these spaces. As we are generally interested in signals which are actual functions rather than distributions we will continue by exploring properties of the Fourier Transform in $\mathcal{L}^p$ spaces for $p \in [1, \infty)$ which is in general a distribution. Later, the extended Paley-Wiener theorem shows that any distribution with compact support (a natural assumption for real-world signals), which is e.g. a function in $\mathcal{L}^p$ on compact sets for $p \in [1, \infty]$, result in entire functions of exponential type and vice versa.

Before we start by introducing the specific spaces in distribution theory, we want to define some notations which are used throughout this work.

2.1 Spaces

2.1.1 Notations

As usual, ℓ^p is the space of all infinite sequences $x = \{x_n\}_{n\in\mathbb{Z}}$ with the norm $\|x\|_{\ell^p} = \left(\sum_{n\in\mathbb{Z}} |x_n|^p\right)^{1/p}$, and $\mathbb{C}^N$ stands for the N-dimensional Euclidean space with the inner product $\langle x, y\rangle = y^* x = \sum_{n=1}^{N} x_n \overline{y_n}$ where y^* is the conjugate transpose of $y \in \mathbb{C}^N$. Note that for sequences we will sometimes also use the equivalent notations $\{x_n\}_{n\in\mathbb{Z}} = \{x_n\}_n = \{x_n\}$ for convenience.

Now let Ω be an open set in $\mathbb{R}^n$. We denote by $C_c(\Omega)$ the space of continuous functions with compact support in Ω. It is equipped with the supremum norm

$\|f\|_\infty := \sup_{x\in\Omega} |f(x)|$. Note that although it is a normed space, it is not complete as it is e.g. dense in $\mathcal{L}^p(\Omega)$ for $p \in [1,\infty)$ (see [35] Theorem 3.14.). $C^k(\Omega)$ with $k \in [1,\infty]$ includes all continuous functions for which the partial derivatives up to order $\leq k$ all exist and are continuous. Functions which are continuous and vanish at infinity are denoted by $C_0(\Omega)$ and this space in turn is a Banach space.

Furthermore we define the Lebesgue spaces $\mathcal{L}^p(\Omega)$ for $p \in [1,\infty]$. In the case where $p \in [1,\infty)$, $\mathcal{L}^p(\Omega)$ includes all measurable functions from Ω to $\mathbb{R}$ fulfilling the inequality condition

$$\|f\|_p := \int_\Omega |f(x)|^p \mathrm{d}x < \infty\,. \tag{2.1}$$

For $p = \infty$ we have instead

$$\|f\|_\infty := \sup_{x\in\Omega} |f(x)| < \infty\,. \tag{2.2}$$

$\|f\|_p$ is also referred to as the norm in $\mathcal{L}^p(\Omega)$. All Lebesgue spaces are Banach spaces. As special cases, $\mathcal{L}^2(\Omega)$ is a Hilbert space (complete inner product space) and $\mathcal{L}^\infty(\Omega)$ is not separable. This results in $\mathcal{L}^2(\Omega)$ being easiest to deal with, while $\mathcal{L}^\infty(\Omega)$ is the least tractable. The inner products in $\mathcal{L}^2(\Omega)$ are induced by the corresponding norm using the polarization identity

$$\langle f, g\rangle = \left\|\frac{f+g}{2}\right\|^2 + \left\|\frac{f-g}{2}\right\|^2 \tag{2.3}$$

and are specifically given by

$$\langle f, g\rangle = \int_\Omega \overline{f(x)}g(x)\mathrm{d}x \;\text{ for } f, g \in \mathcal{L}^2\,. \tag{2.4}$$

Another important function space is $\mathcal{L}^1_{loc}(\Omega)$ which includes all functions which are absolutely integrable on all compact sets in $\mathbb{R}$, i.e.

$$\int_\mathbb{K} |f(x)|\mathrm{d}x < \infty \tag{2.5}$$

for all compact $\mathbb{K} \subset \Omega$.

Partial differentiation of a function will be denoted by $D^\alpha = \left(-\mathrm{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \ldots \left(-\mathrm{i}\frac{\partial}{\partial x_n}\right)^{\alpha_n}$ where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index in $\mathbb{N}_0^n$ and n is the dimension of the space the function is defined on (in this work we usually have $n = 1$). Moreover we write $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

2.1.2 Test functions and Schwartz spaces

Functions in $C_c^\infty(\Omega)$ with compact support are also called test functions $f \in C_c^\infty(\Omega)$ implies $f \in \mathcal{D}(\Omega)$. Other than the space of continuous functions however, referring to $\mathcal{D}(\Omega)$ implies a different metric than the supremum norm.

Definition 2.1 (Test function space)**.** Define compact sets K_i such that $\Omega = \cup_{i\in\mathbb{N}} K_i$ and $K_i \subset K_{i+1}$. We denote by $\mathcal{D}(\Omega)$ the function space $C_c^\infty(\Omega)$ with the seminorms

$$\|\phi\|_i = \sup_{\substack{x\in K_i \\ |\alpha|\le i}} |D^\alpha \phi(x)| \tag{2.6}$$

for each $\phi \in \mathcal{D}(\Omega)$ and multi-indices $\alpha \in \mathbb{N}_0^n$.

The locally convex topology of $\mathcal{D}$ is induced by these semi-norms and we say that $\phi_k \to \phi,\ k \to \infty$ in $\mathcal{D}$ whenever

$$D^\alpha \phi_k(x) \to D^\alpha \phi(x), \qquad k \to \infty \tag{2.7}$$

uniformly for every multi-index α and there is a fixed compact subset of Ω containing the supports of all ϕ_k.

By extending the space $\mathcal{D}(\Omega)$ to functions with infinite support we obtain the Schwartz spaces $\mathcal{S}(\Omega)$.

Definition 2.2 (Schwartz space)**.** By the Schwartz space $\mathcal{S}(\Omega)$ we denote the set of all functions $\phi \in C^\infty$ for which the following holds

$$\|\phi\|_{\alpha,\beta} = \sup_{x\in\Omega} |x^\beta D^\alpha \phi(x)| < \infty \qquad \forall \alpha, \beta \in \mathbb{N}_0^n \tag{2.8}$$

The left hand side with different α, β are called semi-norms and define a weak topology in $\mathcal{S}$.

As in $\mathcal{D}$, convergence $\phi_k \to \phi,\ k \to \infty$ is given in $\mathcal{S}$ if

$$\|\phi_k - \phi\|_{\alpha,\beta} = \sup_{x\in\Omega} |x^\beta D^\alpha \phi_k(x) - x^\beta D^\alpha \phi(x)| \to 0, \qquad k \to \infty\,. \tag{2.9}$$

Furthermore, $\mathcal{S}$ is sometimes called the space of rapidly decreasing functions.

Note that $\mathcal{D}$ and $\mathcal{S}$ are vector spaces and although these semi-norms do not define a norm, the spaces are metrizable. For example in $\mathcal{S}$, the countable family of semi-norms $\|\cdot\|_{\alpha,\beta}$ can be used to define the metric

$$d(f,g) = \sum_{\alpha,\beta\in\mathbb{N}_0^n} \frac{c_{\alpha,\beta}\|f-g\|_{\alpha,\beta}}{1+\|f-g\|_{\alpha,\beta}} \tag{2.10}$$

where $c_{\alpha,\beta}$ are positive constants such that $\sum_{\alpha,\beta\in\mathbb{N}_0^n} c_{\alpha,\beta}$ converges.

2.1.3 Distributions and tempered distributions

In this section we will introduce continuous linear functionals on both $\mathcal{D}$ and $\mathcal{S}$ which are called distributions.

Definition 2.3 (Distributions)**.** A distribution u in Ω is a linear functional on $\mathcal{D}(\Omega)$ such that to every compact set $\mathbb{K} \subset \Omega$ there exist constants C and k such that

$$|u(\phi)| \leq C \sum_{|\alpha| \leq k} \sup_{x \in \mathbb{K}} |D^\alpha \phi(x)|, \; \forall \phi \in C_c^\infty(\mathbb{K}) \tag{2.11}$$

The space of all such functionals is denoted by $\mathcal{D}'(\Omega)$.

For fixed $\phi \in \mathcal{D}$ we call $\|u\|_\phi := |u(\phi)|$ semi-norms which define the weak topology of $\mathcal{D}'$. Convergence $u_j \to u$ in $\mathcal{D}'$ means that

$$u_j(\phi) \to u(\phi) \qquad \forall \phi \in \mathcal{D}\,. \tag{2.12}$$

In the following we will omit the domain Ω. Note that if you consider the right hand side as the "norm" of ϕ, this condition of boundedness corresponds to the characterization of a linear operator being (sequentially) continuous. The following theorem formally establishes the analogy

Theorem 2.4. *A linear functional u is a distribution if and only if $u(\phi_k) \to 0$ when $k \to \infty$ for every sequence $\phi_k \to 0$ in $\mathcal{D}$*

Proof. See [18] Theorem 1.3.1. □

The following corollary is then an immediate consequence.

Corollary 2.5. *The space of distributions $\mathcal{D}'$ is the dual space of the space of test functions $\mathcal{D}$ with its locally convex topology, i.e. it consists of the linear bounded functionals on $\mathcal{D}$.*

Note that each function $\tilde{u} \in \mathcal{L}^1_{loc}$ can be identified with a distribution $u \in \mathcal{D}'$ by the linear form

$$u(\phi) = \int_\Omega \phi(x)\tilde{u}(x)\mathrm{d}x \tag{2.13}$$

Definition 2.6 (Support)**.** The complement of the support of u, denoted by $\operatorname{supp} u$, is the largest open subset $A \subset \Omega$ where $u(\phi) = 0$ whenever $\operatorname{supp} \phi = A$.

Remark. When u is a continuous function, this definition coincides with the usual notion of support, which is the set on which the function is nonzero.

Definition 2.7 (Convolution)**.** If $u \in \mathcal{D}'$ and $\phi \in \mathcal{D}$, we define the "convolution" $u * \phi$ as

$$(u * \phi)(x) = u_y(\phi(x - y)).$$

For Fourier Transforms, a specific class of distributions is of particular interest.

Definition 2.8 (Tempered distributions)**.** A continuous linear functional u on $\mathcal{S}$ is called a tempered (or temperate) distribution. The space of all tempered distributions is denoted by $\mathcal{S}'$ since it is by definition the dual space of $\mathcal{S}$. The weak topology on $\mathcal{S}'$ is defined by the semi-norms $|u(\phi)|$ with fixed $\phi \in \mathcal{S}$ similar to the space of distributions $\mathcal{D}'$.

Definition 2.9 (Regular distributions)**.** Whenever $u \in \mathcal{S}'$ can be identified with a function $u \in \mathcal{L}^1_{loc}$, we call u a regular distribution.

Remark. Note that the whole space $\mathcal{L}^1_{loc}$ belongs to $\mathcal{D}'$ however only $\mathcal{L}^1_{loc} \bigcap \mathcal{S}'$ are regular distributions.

In the following the simplifying notation $X \subset \mathcal{S}'$ only denotes the inclusion of the sets rather than the entire spaces with their respective topologies. Thus it only means that functions in X can be identified with a tempered distribution while the spaces themselves are not "subspaces" of one another since they can have different norms and metrics.

2.1.4 Theorems of denseness

For the newly introduced spaces in Section 2.1.3, several theorems of denseness and inclusions hold.

Theorem 2.10. *$\mathcal{D}$ is dense in $\mathcal{L}^p$, dense in C_0 and dense in $\mathcal{S}$ in the respective topologies.*

Proof. For the first statement see [18] Theorem 1.2.1. Denseness in C_0 with respect to the supremum norm follows using fundamental functional analytic methods. It remains to show that $\mathcal{D}$ is dense in $\mathcal{S}$. First of all it is clear that $\mathcal{D}$ is a subspace in $\mathcal{S}$. Now let $\phi \in \mathcal{S}$ and take $\psi \in \mathcal{D}$ such that $\psi(x) = 1$ when $|x| \leq 1$. Define $\phi_\epsilon(x) = \phi(x)\psi(\epsilon x)$. Now it is easy to see that $\phi_\epsilon \in \mathcal{D}$ is a sequence converging to ϕ in $\mathcal{S}$ when $\epsilon \to 0$ and the proof is complete. $\square$

For two linear spaces $X \subset Y$, let's define the inclusion map

$$\begin{aligned} i : & X \to Y \\ & x \mapsto x\,. \end{aligned}$$

Whenever i is continuous, we also say X is continuously embedded in Y, i.e. $\|x\|_Y \leq C\|x\|_X$, we know for the dual spaces that $Y' \subset X'$ also with a continuous inclusion map i^T. If moreover $i(X)$ is dense in Y, then i^T is injective. Since furthermore $\mathcal{D} \subset \mathcal{S}$ and it follows from the Definitions 2.3 and 2.8 that convergence in $\mathcal{D}$ implies convergence in $\mathcal{S}$. Thus $\mathcal{D}$ is continuously embedded in $\mathcal{S}$ and we have

$$\mathcal{D} \subset \mathcal{S};\ \mathcal{S}' \subset \mathcal{D}'\,. \tag{2.14}$$

The more interesting inclusions

$$\mathcal{L}^p \subset \mathcal{L}^1_{loc} \subset \mathcal{D}' \tag{2.15}$$

for $p \in [1, \infty)$ also follow naturally, since on compact sets $\mathbb{K}$ we have $\int_{\mathbb{K}} |f(x)|^p < \infty \implies \int_{\mathbb{K}} |f(x)|^q < \infty$ for $p > q$ as we only have to consider the improper integrals at singularities and do not have to investigate how the integral behaves in infinity. Thus $\mathcal{L}^1_{loc}$ is the biggest space. The second inclusion follows from (2.13).

Theorem 2.11. *$\mathcal{L}^1 \bigcap \mathcal{L}^p$ is dense in $\mathcal{L}^p$ for $p \in [1, \infty)$*

Proof. The proof is rather straight forward. We know that $C_c^\infty = \mathcal{D}$ is dense in both $\mathcal{L}^1$ and $\mathcal{L}^p$. Thus to every $f \in \mathcal{L}^1 \cap \mathcal{L}^p$ there exists a sequence $\{f_n\} \subset C_c^\infty \subset \mathcal{L}^1 \bigcap \mathcal{L}^p$ for which $f_n \to f$ for $n \to \infty$ in $\mathcal{L}^p$. Now because also $\mathcal{L}^1 \bigcap \mathcal{L}^p \subset \mathcal{L}^p$, the theorem follows. □

2.2 Fourier Transform

After the introduction of distribution and function spaces we now turn to the Fourier Transform defined thereon. For simplicity we will from now on look at $\Omega \subset \mathbb{R}$ only.

2.2.1 Fourier Transform on $\mathcal{L}^1$

Definition 2.12 (Fourier Transform on $\mathcal{L}^1$)**.** For $f \in \mathcal{L}^1(\mathbb{R})$, the Fourier Transform is defined as

$$\mathcal{F}f(\omega) := \int_{\mathbb{R}} f(t) \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}t \,. \tag{2.16}$$

We will also write $\hat{f} := \mathcal{F}f$.

The absolute convergence of this integral is trivial, so that the mapping is well-defined and we also have

Theorem 2.13 (Inverse Fourier Transform)**.** *For $f \in \mathcal{L}^1$ and $\hat{f} \in \mathcal{L}^1$ as in* (2.16)*, the Inverse Fourier Transform reads*

$$\mathcal{F}^{-1}\hat{f}(t) = f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\omega) \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}\omega \,. \tag{2.17}$$

Proof. See [35] Theorem 9.11. □

The following lemma establishes the relation between the norm of $f \in \mathcal{L}^1$ and the supremum of $\hat{f} \in \mathcal{L}^1$.

Theorem 2.14 (Riemann-Lebesgue Lemma)**.** *For all $f \in \mathcal{L}^1(\mathbb{R})$ we have $\|\hat{f}\|_\infty \leq \|f\|_1$. In particular, $\hat{f} \in C_0(\mathbb{R})$.*

Proof. See [35] Theorem 9.6. □

Note however that $\mathcal{F} : \mathcal{L}^1 \to C_0$ is not surjective.

2.2.2 Fourier Transform on $\mathcal{S}$ and $\mathcal{S}'$

We now see that for $\mathcal{S}, \mathcal{S}'$ the Fourier Transform is also well-defined and bijective.

Theorem 2.15. *The Fourier Transform $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is defined as*

$$\mathcal{F}\phi(\omega) = \int_{\mathbb{R}} \phi(x)\mathrm{e}^{-\mathrm{i}\omega x}\mathrm{d}x \tag{2.18}$$

and is bijective and continuous in the topology of $\mathcal{S}$. The Inverse Fourier Transform reads

$$\mathcal{F}^{-1}\hat{\phi}(x) = \phi(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\phi}(\omega)\,\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega\,.$$

Proof. We closely follow the proof in [18] Lemma 1.7.1. We first need to show that $\hat{\phi} = \mathcal{F}\phi \in \mathcal{S}$, i.e. $\sup_\omega |\omega^\beta D^\alpha \hat{\phi}(\omega)| < \infty$ for all $\alpha, \beta \in \mathbb{N}_0$. Differentiation under the integral yields

$$D^\alpha \hat{\phi}(\omega) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}x\omega}(-x)^\alpha \phi(x)\mathrm{d}x\,.$$

Hence we have that $\hat{\phi} \in C^\infty$. Integrating by parts we also obtain

$$\omega^\beta D^\alpha \hat{\phi}(\omega) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}x\omega} D^\beta((-x)^\alpha \phi(x))\mathrm{d}x\,.$$

This equality is mainly based on the fact that $\phi \in \mathcal{S}$ so that for all α, β we have $[\mathrm{e}^{-\mathrm{i}x\omega} D^\beta((-x)^\alpha\phi(x))]_{-\infty}^{\infty} = 0$. Now because $D^\beta((-x)^\alpha\phi(x)) \in \mathcal{S} \subset \mathcal{L}^1$, it follows from Theorem 2.14 that $\omega^\beta D^\alpha\hat{\phi}(\omega)$ is bounded so that $\hat{\phi} \in \mathcal{S}$.

In the next step we need to prove continuity (equivalent to sequential continuity). For this we observe that for any $\phi_k \to \phi$

$$\begin{aligned} |\omega^\beta D^\alpha \mathcal{F}[\phi_k - \phi](\omega)| &\leq \int_{\mathbb{R}} |D^\beta((-x)^\alpha[\phi_k - \phi](x))|\mathrm{d}x \qquad (2.19)\\ &\leq \sup_{x\in\mathbb{R}} |D^\beta((-x)^\alpha[\phi_k - \phi](x))|(1+|x|)^2 \int_{\mathbb{R}} \frac{1}{(1+|x|)^2}\mathrm{d}x\,. \end{aligned}$$

Because $\mathcal{F}\phi_k, \mathcal{F}\phi \in \mathcal{S}$, by (2.8) this supremum will also be bounded and by (2.9) we have $|\omega^\beta D^\alpha \mathcal{F}[\phi_k - \phi](\omega)| \to 0$ as $k \to \infty$. For the Inverse Fourier Transform see the proof of Theorem 1.7.1 in [18]. □

Definition 2.16 (Fourier Transform on $\mathcal{S}'$)**.** The Fourier Transform for $u \in \mathcal{S}'$ is defined as

$$\hat{u}(\phi) = u(\hat{\phi}) \qquad \forall\phi \in \mathcal{S}\,. \tag{2.20}$$

Theorem 2.17. *The mapping $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$ is bijective and continuous.*

Proof. First, define $\check{\phi}(x) = \phi(-x)$ for $\phi \in \mathcal{S}$ and $\check{u}(\phi) = u(\check{\phi})$ for $u \in \mathcal{S}'$. The Fourier Transform is bijective because we have $\hat{\hat{u}} = u(\hat{\hat{\phi}}) = 2\pi u(\check{\phi}) = 2\pi \check{u}(\phi)$ by the inversion formula $\hat{\hat{\phi}} = 2\pi\check{\phi}$ for $\phi \in \mathcal{S}$ (see proof in [18] Theorem 1.7.1.). In order to show continuity let $u_k \to u$ in $\mathcal{S}'$ as $k \to \infty$. Then for all $\phi \in \mathcal{S}$ Definition 2.16 yields

$$\mathcal{F}u_k(\phi) = u_k(\mathcal{F}\phi) \to u(\mathcal{F}\phi) = \mathcal{F}u(\phi), \text{ for } k \to \infty\,.$$

□

2.2.3 Fourier Transform on $\mathcal{L}^p$

After the straight-forward definition of Fourier Transforms in $\mathcal{L}^1, \mathcal{S}$ and $\mathcal{S}'$, it is now of our interest to examine whether it is also possible to extend it to $\mathcal{L}^p$ spaces with $p \neq 1$. We start with a theorem on the Fourier Transform on the Hilbert space $\mathcal{L}^2$.

Theorem 2.18 (Fourier Transform on $\mathcal{L}^2$)**.** *Let $f \in \mathcal{L}^1 \bigcap \mathcal{L}^2$. Then*
(i) $\hat{f} \in \mathcal{L}^2$ as defined in (2.16) *and the Plancherel formula holds $\|f\|_2 = \|\hat{f}\|_2$.*
(ii) For every $f \in \mathcal{L}^2$ we also have $\|f\|_2 = \|\hat{f}\|_2$.
(iii) $\mathcal{F} : \mathcal{L}^1 \bigcap \mathcal{L}^2 \to \mathcal{L}^2$ can be uniquely extended to a bounded isometry $\mathcal{F} : \mathcal{L}^2 \to \mathcal{L}^2$.
(iv) The Fourier Transform $\mathcal{F} : \mathcal{L}^2 \to \mathcal{L}^2$ is unitary and we can define an inverse in the following sense

$$\| \int_{-n}^{n} \hat{f}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega - \mathcal{F}^{-1}\hat{f}\|_2 \to 0 \qquad \textit{for } A \to \infty \tag{2.21}$$

Proof. This proof mainly follows [12]. Note that other than in our definition, the paper uses the norming constant $1/\sqrt{2\pi}$ in front of the the integrals in both the Fourier Transform and the inverse formula. Let $f \in \mathcal{L}^1 \cap \mathcal{L}^2$ be arbitrary. Then by Theorem 2.14 we have $\hat{f} \in \mathcal{L}^\infty$ so that for any $\epsilon > 0$, the following holds

$$\int_{\mathbb{R}} |\hat{f}(\omega)|^2 \mathrm{e}^{-\epsilon\omega^2}\mathrm{d}\omega < \infty \tag{2.22}$$

By definition and absolute integrability of the integrand (Fubini), we obtain further

$$\begin{aligned}\int_{\mathbb{R}} |\hat{f}(\omega)|^2 \mathrm{e}^{-\epsilon\omega^2}\mathrm{d}\omega &= \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{f(x)} f(y) j_\epsilon(x-y)\mathrm{d}x\mathrm{d}y \\ &= \langle f, f * j_\epsilon\rangle_{\mathcal{L}^2} \end{aligned}\tag{2.23}$$

where

$$j_\epsilon(x-y) := \frac{1}{\sqrt{4\epsilon\pi}}\mathrm{e}^{-\frac{(x-y)^2}{4\epsilon}}$$

using a Gaussian function. Now we can use Theorem 2.16 in [24] (where $j(x) = \mathrm{e}^{-(x-y)^2\pi}$) from which it follows that $f * j_\epsilon \to f$ in $\mathcal{L}^2$ and the right hand side of (2.23) goes to $\|f\|^2$ as $\epsilon \to 0$ because the inner product is continuous. In particular, $\langle f, f * j_\epsilon\rangle_{\mathcal{L}^2}$ is uniformly bounded in ϵ. Since we also have

$g_\epsilon(\omega) := |\hat{f}(\omega)|^2 \mathrm{e}^{-\epsilon\omega^2} \leq g_{\epsilon'}(\omega)\ \forall \epsilon' < \epsilon$, by monotone convergence we can exchange the limit with the integral such that

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}} |\hat{f}(\omega)|^2 \mathrm{e}^{-\epsilon\omega^2} \mathrm{d}\omega = \int_{\mathbb{R}} |\hat{f}(\omega)|^2 \lim_{\epsilon\to 0} \mathrm{e}^{-\epsilon\omega^2} \mathrm{d}\omega = \|\hat{f}\|_2$$

and (i) is proved.

We can extend the Fourier Transform to the bigger domain $\mathcal{L}^2$ uniquely by the completeness of $\mathcal{L}^2$. Because $\mathcal{L}^1 \bigcap \mathcal{L}^2$ is dense in $\mathcal{L}^2$ by Theorem 2.11, for any $f \in \mathcal{L}^2$ we can find an approximating sequence $f_n \in \mathcal{L}^1 \bigcap \mathcal{L}^2$ with $f_n \to f$ in $\mathcal{L}^2$. By (i), we then know that $\hat{f}_n$ is also a Cauchy sequence in $\mathcal{L}^2$ and we can define $\hat{f}$ as the limit of $\hat{f}_n$. The limit does not depend on the particular choice of the sequence since for two different $f_n \to f$ and $\tilde{f}_n \to f$ we have by (i) and the triangular inequality $\|\hat{f}_n - \hat{\tilde{f}}_n\|_2 = \|f_n - \tilde{f}_n\|_2 \leq \|f_n - f\|_2 + \|\tilde{f}_n - f\|_2 \to 0$. Then we can use $\lim_{n\to\infty} \|\|\hat{f}\| - \|\hat{f}_n\|\|_2 \leq \lim_{n\to\infty} \|\hat{f} - \hat{f}_n\|_2 = 0$ to obtain

$$\|\hat{f}\| = \lim_{n\to\infty} \|\hat{f}_n\| = \lim_{n\to\infty} \|f_n\| = \|f\|$$

and (ii) is proved. One possible definition for the Fourier Transform of a function $f \in \mathcal{L}^2$ can be given as

$$\hat{f}(\omega) = \lim_{\epsilon\to 0} f(x) \mathrm{e}^{\mathrm{i}\omega x} \mathrm{e}^{-\epsilon x^2} \mathrm{d}x \,. \tag{2.24}$$

Isometry in (iii) can be shown by the polarization identity for Hilbert spaces and injectivity in (iv) is given by isometry. For surjectivity one can use a result in the proof for 9.13 in [35], where it was proven $\mathcal{F}(\mathcal{L}^1 \bigcap \mathcal{L}^2) =: Y$ is dense in $\mathcal{L}^2$. By (iii) for each $y \in \mathcal{L}^2$ we have one x such that $y = \mathcal{F}x$. By denseness one can then find $y_n \in Y$ such that $y_n \to y$. For each y_n we can apply (**??**) to obtain $x_n \in \mathcal{L}^1 \cap \mathcal{L}^2$ which satisfies $y_n = \mathcal{F}x_n$. Since $\mathcal{F}$ is isometric, $\|\mathcal{F}x_n - \mathcal{F}x\| \to 0$ implies $\|x_n - x\| \to 0$. If we now choose $x_n = \chi_{[-n,n]}x$ (2.21) follows and the proof is complete. For more detailed explanations see [35] Theorem 9.13 or [12]. □

Definition 2.19 (Conjugate exponent)**.** The conjugate exponent p' of p fulfills the equation

$$\frac{1}{p'} + \frac{1}{p} = 1 \tag{2.25}$$

We now consider $\mathcal{L}^p$ with $p \neq 2$. The question is mainly when $\mathcal{F} : \mathcal{L}^{p'} \to \mathcal{L}^p$ is bounded and surjective. The famous Riesz-Thorin theorem (see [32] Vol. 2., Theorem IX.17) states that if for a linear map $T : \mathcal{L}^{p_0} + \mathcal{L}^{p_1} \to \mathcal{L}^{p_0'} + \mathcal{L}^{p_1'}$, both $T : \mathcal{L}^{p_0} \to \mathcal{L}^{p_0'}$ and $T : \mathcal{L}^{p_1} \to \mathcal{L}^{p_1'}$ are bounded, then $T : \mathcal{L}^p \to \mathcal{L}^{p'}$ is also bounded for $p \in [p_0, p_1]$. Here, p_i and p_i' are conjugate exponents. One can immediately obtain the Hausdorff-Young inequality for $p_0 = 1$ and $p_1 = 2$ with Theorems 2.14 and 2.18.

Theorem 2.20 (Hausdorff-Young)**.** *Let $1 \leq p' \leq 2$. The Fourier Transform $\mathcal{F} : \mathcal{L}^{p'} \to \mathcal{L}^p$ is a bounded linear map and*

$$\|\mathcal{F}f\|_p \leq \|f\|_{p'} \tag{2.26}$$

Remark. It is important to note that by Theorem 2.17 and $\mathcal{L}^p \subset \mathcal{S}'$ for $1 \leq p < \infty$ the mapping $\mathcal{F} : \mathcal{L}^{p'} \to \mathcal{L}^p$ is injective. However it is not surjective and there are functions $f \in \mathcal{L}^p$ with $p > 2$, for which the (Inverse) Fourier Transform can not be identified with a function in $\mathcal{L}^{p'}(\mathbb{R})$ (see Fig. 2.1 (a)). For $1 \leq p \leq 2$ however, we can induce from this theorem that for any $f \in \mathcal{L}^p$, $\mathcal{F}^{-1}f \in \mathcal{L}^{p'}$ since by (2.26) the mapping $\mathcal{F}_r : \mathcal{L}^{p'} \to \mathcal{F}(\mathcal{L}^{p'}) \subset \mathcal{L}^p$ with $1 \leq p' \leq 2$ is bijective (see Fig. 2.1 (b)). Also note that for regular distributions $\mathcal{L}^1_{loc} \cap \mathcal{S}' \supset \mathcal{L}^p$ in general.

2.2.4 Fourier Laplace transform

Besides Lebesgue spaces with $1 \leq p \leq 2$, the Fourier Transform of distributions with compact support also yield regular distributions. Before we come to the main theorem, we need to look at the extension of the domain of the functionals $u \in \mathcal{D}'$ to the space of continuous functions with infinite support C^∞. Let us denote by $\mathcal{E}'$ the distributions $u \in \mathcal{D}'$ with compact support. Whenever we have a regular distribution $u \in \mathcal{E}'$, it is easy to extend the domain of u to C^∞ since $u(\phi) = \int_{\mathbb{R}} u\phi \mathrm{d}x$ is well-defined. For general distributions we have the following result:

Theorem 2.21. *Let $u \in \mathcal{E}'$, so that $\operatorname{supp} u \bigcap \operatorname{supp} \phi$ is compact for all $\phi \in \mathbb{C}^\infty$. Then there is one and only one linear form $\tilde{u}$ on C^∞ such that*
(i) $\tilde{u}(\phi) = u(\phi)$ if $\phi \in \mathcal{D}$
(ii) $\tilde{u}(\phi) = 0$ if $\phi \in C^\infty$ and $\operatorname{supp} u \bigcap \operatorname{supp} \phi = \emptyset$.

Remark. Note that $\hat{u} \in \mathbb{C}^\infty$ since $u = \psi u$ with indicator function $\psi \in \mathcal{D}$ nonzero on $\operatorname{supp} u$ and thus$\hat{u} = \hat{u} * \hat{\psi}$ (see [36] Theorem 7.19 (a), (e)).

Proof. See [18] Theorem 1.5.1. □

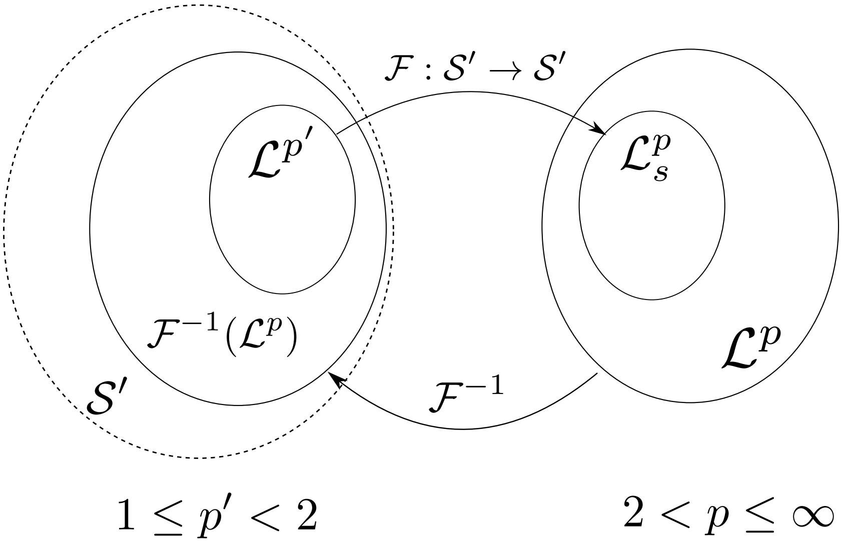
Decompose each $\phi \in C^\infty$ using $\psi \in C_c^\infty$ with $\psi = 1$ in a neighborhood of $\operatorname{supp} u$ such that $\phi = \phi_0 + \phi_1$ where

$$\phi_0 = \psi\phi \in C_c^\infty$$
$$\phi_1 = (1 - \psi)\phi$$

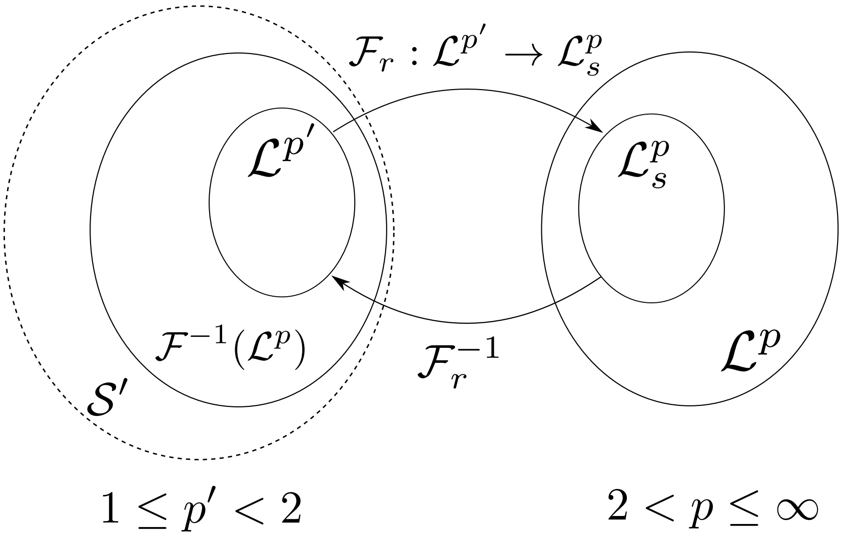
such that $\operatorname{supp} u \bigcap \operatorname{supp} \phi_1 = \emptyset$. (ii) and linearity of the distributions then imply that

$$u(\phi) = u(\psi\phi) + u((1 - \psi)\phi) = u(\psi\phi) \qquad \forall \phi \in C^\infty \, . \tag{2.27}$$

Now we can proceed to define the Fourier-Laplace transform which we will use throughout the following chapters



(a)



(b)

Figure 2.1: (a) The Fourier Transform is bijective on the space of tempered distributions. This illustration shows how the range of the Inverse Fourier Transform on $\mathcal{L}^p$ with $2 < p \leq \infty$ also contains, apart from functions in $\mathcal{L}^{p'}$ with $1 \leq p' < 2$, tempered distributions in general. (b) The restricted Fourier Transform from $\mathcal{L}^{p'}$ to the range $\mathcal{F}(\mathcal{L}^{p'}) =: \mathcal{L}^p_s \subset \mathcal{L}^p$ is invertible. Therefore the range of the Inverse Fourier Transform on $\mathcal{L}^{p'}$ with $1 \leq p' < 2$ only contains functions in $\mathcal{L}^p$ with $2 < p \leq \infty$.

Theorem 2.22 (Fourier-Laplace Transform)**.** *The Fourier Transform of a distribution $u \in \mathcal{E}'$ is the function*

$$\hat{u}(\xi) = u_x(\mathrm{e}^{-\mathrm{i}x\xi}). \tag{2.28}$$

The right-hand side is also defined for every complex number $\xi \in \mathbb{C}$ and is an entire analytic function of ξ called the Fourier-Laplace transform of u.

Proof. The proof reproduces the one in [18] to Theorem 1.7.5. If u is a function, we can simply use Fubini's Theorem to obtain the result, since the integrand is in $\mathcal{L}^1$:

$$\begin{aligned}\hat{u}(\phi) = u(\hat{\phi}) &= \int u(x) \int \phi(\xi)\mathrm{e}^{-\mathrm{i}x\xi}\mathrm{d}\xi\mathrm{d}x \\ &= \int \phi(\xi) \int u(x)\mathrm{e}^{-\mathrm{i}x\xi}\mathrm{d}x\mathrm{d}\xi \overset{!}{=} \int \phi(\xi)\hat{u}(\xi)\mathrm{d}\xi\end{aligned}$$

and thus

$$\hat{u}(\xi) = u(\mathrm{e}^{-\mathrm{i}x\xi})\,. \tag{2.29}$$

To prove it in general, we again use a mollifier $\phi_\epsilon(x) = \epsilon^{-1}\phi(x/\epsilon)$ with $\phi \in C_c^\infty$. For these mollifiers we know by [18] Theorem 1.6.3. that $u * \phi_\epsilon \to u$ in the weak topology of $\mathcal{S}'$ so that by Definition 2.16 it follows that $\mathcal{F}(u * \phi_\epsilon) \to \mathcal{F}u$ in $\mathcal{S}'$ for $\epsilon \to 0$. Furthermore, $(u * \phi_\epsilon) \in \mathbb{C}^\infty$ (see [18] Theorem 1.6.1.) and applying (2.29) we have $\widehat{(u * \phi_\epsilon)} = (u * \phi_\epsilon)(\mathrm{e}^{-\mathrm{i}x\xi})$. Using Theorem 2.21 we can now define the following analytic function

$$(u * \phi_\epsilon)(\mathrm{e}^{-\mathrm{i}x\xi}) = u(\check{\phi}_\epsilon * \mathrm{e}^{-\mathrm{i}x\xi}) = \hat{\phi}(\epsilon\xi)u(\mathrm{e}^{-\mathrm{i}x\xi}). \tag{2.30}$$

with $\check{\phi}(x) = \phi(-x)$. It is analytic because $\mathrm{e}^{-\mathrm{i}x\xi}$ is continuously differentiable in ξ and thus we can exchange the differentiation and the integral. The first equality follows from $u(\psi) = (u * \check{\psi})(0)$ for $\psi \in \mathcal{D}$ by definition (2.7) and [18] Theorem 1.6.2. stating $(u * \phi) * \psi = u * (\phi * \psi)$ for any $\phi, \psi \in \mathcal{D}$ and $u \in \mathcal{D}'$. Theorem 1.6.2. can be applied since by Theorem 2.21 we have $u(\psi) = u(\tilde{\psi})$ for $\tilde{\psi}(x) = \mathrm{e}^{-\mathrm{i}x\xi}\chi_{\operatorname{supp} u}(x)$ where χ is the indicator function. Additionally, $\hat{\phi}(\epsilon\xi) \to \hat{\phi}(0) = 1$ uniformly on every compact subset of $\mathbb{C}$ when $\epsilon \to 0$. As $(u * \phi_\epsilon)(\mathrm{e}^{-\mathrm{i}x\xi})$ is entire (nicely shown in the proof of [36] Theorem 7.23.) and $(u * \phi_\epsilon)(\mathrm{e}^{-\mathrm{i}x\xi}) \to u(\mathrm{e}^{-\mathrm{i}x\xi})$ uniformly on compact sets, it is allowed to exchange the limit and the integral (see Weierstrass' convergence theorem e.g. [33] Theorem 8.§4.1.) from which we that $u(\mathrm{e}^{-\mathrm{i}x\xi})$ is also an entire function of ξ. Since we know at the same time that $\widehat{(u * \phi_\epsilon)} = (u * \phi_\epsilon)(\mathrm{e}^{-\mathrm{i}x\xi}) \to \hat{u}$ in $\mathcal{S}'$ for $\xi \in \mathbb{R}$, we have shown that the restriction of $u(\mathrm{e}^{-\mathrm{i}x\xi})$ to real arguments is the Fourier Transform of u, while for $\xi \in \mathbb{C}$, $u(\mathrm{e}^{-\mathrm{i}x\xi})$ is an entire function. □

2.2.5 Paley-Wiener-Theorems

We now show that every entire function of exponential type $\leq \sigma$ is the Fourier-Laplace Transform of a (tempered) distribution with compact support $[-\sigma, \sigma]$. These results enable us to discuss interpolation in the Bernstein spaces rather than in the original space of signals with compact support and transfer the results back via the Fourier-Laplace Transform.

Theorem 2.23 (Extended Paley-Wiener-Theorem)**.** *An entire analytic function $U(z)$ is the Fourier-Laplace transform of a distribution with support in $[-\sigma, \sigma]$ if and only if for some constants C and N we have*

$$|U(z)| \leq C(1 + |z|)^N \mathrm{e}^{\sigma |\operatorname{Im} z|} \,. \tag{2.31}$$

Proof. The proof can be found in [18] for Theorem 1.7.7. The necessity follows somewhat easily. The sufficiency uses the result that U is the Fourier-Laplace transform of a function $u \in C_c^\infty([-\sigma, \sigma])$ if and only if for every integer N there exists a constant C_N such that

$$|U(z)| \leq C_N (1 + |z|)^{-N} \mathrm{e}^{\sigma |\operatorname{Im} z|} \,.$$

□

Remark. By this theorem we know that the Fourier-Laplace transform from distributions with finite support $[-\sigma, \sigma]$ to the entire functions of exponential type σ is bijective. Surjectivity directly follows from the theorem and the bijectivity of the Fourier Transform on the tempered distributions yield injectivity. Since according to 2.22, the Fourier-Laplace Transform is equivalent to the Fourier Tranform for $z \in \mathbb{R}$, the inverse Fourier-Laplace Transform is identical to (2.15) so that for every regular distribution $u \in \mathcal{L}^1(\mathbb{T})$ we have

$$u(t) = \int_{\mathbb{R}} U(\omega) \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}\omega \,. \tag{2.32}$$

In the case of $U \in \mathcal{B}^2_{T/2}$ we have an even more specific result yielding regular distributions in $\mathcal{L}^2([-\sigma, \sigma])$.

Theorem 2.24 (Paley-Wiener Theorem)**.** *An entire function $U(z)$ is of exponential type σ and $\int_{\mathbb{R}} |U(x)|^2 \mathrm{d}x < \infty$ if and only if there exists an $u \in \mathcal{L}^2([-\sigma, \sigma])$ such that*

$$U(z) = \int_{-\sigma}^{\sigma} u(t) \mathrm{e}^{-\mathrm{i}zt} \mathrm{d}t \tag{2.33}$$

for all $z \in \mathbb{C}$.

Proof. Sufficiency follows directly from

$$|U(z)| \leq \int_{-\sigma}^{\sigma} \mathrm{e}^{-|\operatorname{Im} z| t} |u(t)| \mathrm{d}t \leq \mathrm{e}^{\sigma |\operatorname{Im} z|} \int_{-\sigma}^{\sigma} |u(t)| \mathrm{d}t. \tag{2.34}$$

The proof of the necessity is a bit longer and is nicely presented in [35] Theorem 19.3. □

3

Entire functions theory

3.1 Entire functions

In this chapter, we will introduce the Bernstein spaces which contain a certain type of entire holomorphic functions. These are the signal spaces in which we will perform interpolation when shifting the phase retrieval problem from finite to infinite dimensions. Therefore, the final goal here will be to establish interpolation formulas for all Bernstein spaces for which we will use sine-type functions and the Plancherel Pólya Theorem.

3.1.1 Holomorphic functions

In this section we will briefly recall some fundamental concepts in complex analysis. Starting off with well-known theorems for holomorphic functions we will show the equivalence of holomorphism and analyticism. Then entire functions are introduced which represent the most general type of functions we will be dealing with in this thesis.

Definition 3.1 (Connected sets and components)**.** A set E is called not connected if E is the union of two nonempty sets A and B such that

$$\bar{A} \cap B = \emptyset = A \cap \bar{B}$$

where $\bar{A}$ denotes the closure of A. The union of all connected subsets of E that contain a certain $x \in E$ is connected and called a maximally connected subset of E. All these sets for different $x \in E$ are also called components of E.

In the following, if not otherwise noted, Ω will be a region, i.e. a nonempty connected open subset of the complex plane.

Definition 3.2 (Holomorphic)**.** Let $\Omega \subset \mathbb{C}$ and $f : \Omega \to \mathbb{C}$. If

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists for every $z_0 \in \Omega$, then f is called holomorphic in Ω. In the following we will also write $f \in \mathrm{H}(\Omega)$.

Remark (Cauchy-Riemann). It is well-known that a complex function is differentiable in every point if and only if partial derivatives of f exists at every point $x \in \Omega$ and the Cauchy-Riemann equations are fulfilled at all $x \in \Omega$, i.e.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \,. \tag{3.1}$$

[35] Section 11.1. gives a very nice intuition on why this is the case. Whenever a function is differentiable, there exists the following representation

$$f(z) = \alpha x + \beta y + \eta(z) z \,, \qquad z = x + \mathrm{i}y \tag{3.2}$$

with $\eta(z) \to 0$ as $z \to 0$ and $\alpha = \frac{\partial f}{\partial x}(z)$ and $\beta = \frac{\partial f}{\partial y}(z)$. By simple calculations (3.2) is equivalent to

$$f(z) = \frac{\alpha - \mathrm{i}\beta}{2} z + \frac{\alpha + \mathrm{i}\beta}{2} \bar{z} + \eta(z) z. \tag{3.3}$$

With

$$\partial := \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right), \ \bar{\partial} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right),$$

we obtain

$$\frac{f(z)}{z} = (\partial f)(0) + (\bar{\partial} f)(0)\frac{\bar{z}}{z} + \eta(z) \ \forall z \neq 0. \tag{3.4}$$

We see that $f(z)/z$ has a limit at 0 if and only if $(\bar{\partial} f)(0) = 0$. If we write $f = u + \mathrm{i}v$ with u, v real, this is equivalent to the Cauchy-Riemann condition in (3.1). For a more rigorous proof see [39] Section 2.15.

3.1.2 Major theorems

In this section, the Cauchy integral formula and the residue theorem are presented. For this we need to introduce the notion of path integrals.

Definition 3.3 (Curves and paths)**.** A curve γ on a topological space X is a mapping $\gamma : [\alpha, \beta] \subset \mathbb{R} \to X$ where $\alpha < \beta$. By γ^* we denote the range of γ, i.e. $\{\gamma(t) : \alpha \leq t \leq \beta\}$. A path is a piecewise continuously differentiable curve, i.e. $\exists \{s_j\}_{j \in J} : \gamma : [s_{j-1}, s_j] \to X$ is continuous for all $j \in J$ with J a finite index set. A path is closed whenever it is also a closed curve, i.e. $\gamma(\alpha) = \gamma(\beta)$.

Definition 3.4 (Index)**.** Let γ be a closed path and Ω the complement of γ^* with respect to the complex plane. Then we define the index of $z \in \Omega$ as follows

$$\mathrm{Ind}_\gamma(z) = \frac{1}{\mathrm{i}2\pi} \int_\gamma \frac{\mathrm{d}\zeta}{\zeta - z} \,. \tag{3.5}$$

It can be shown that the index is equivalent to how often the path winds around the point z. This follows from the proof of the next lemma.

Lemma 3.5. Ind_γ *is always an integer which is constant in each component of* Ω *and 0 in the unbounded component of* Ω*.*

Proof. We reproduce a proof from [35] Theorem 10.10. First of all we show that Ind_γ is an integer for all $z \in \Omega$. Let $[\alpha, \beta]$ be the parameter interval for γ and for fix $z \in \Omega$ we have by (3.5) and $\mathrm{d}\gamma(s) = \gamma'(s)\mathrm{d}s$

$$\mathrm{Ind}_\gamma(z) = \frac{1}{\mathrm{i}2\pi} \int_\alpha^\beta \frac{\gamma'(s)}{\gamma(s) - z} \mathrm{d}s \,. \tag{3.6}$$

Then the statement that Ind_γ is an integer is equivalent to $\phi(\beta) \overset{!}{=} 1$ with

$$\phi(t) = \exp \int_\alpha^t \frac{\gamma'(s)}{\gamma(s) - z} \mathrm{d}s \,. \tag{3.7}$$

Now we have

$$\frac{\phi'(t)}{\phi(t)} = \frac{\gamma'(t)}{\gamma(t) - z}$$

Then we use that $\frac{\phi}{\gamma - z}$ is continuous on $[\alpha, \beta]$ and has a derivate which is zero on $[\alpha, \beta] \setminus S$ where S is a finite subset of $[\alpha, \beta]$ (because γ is a path). Since S is finite, $\frac{\phi}{\gamma - z}$ is a constant on $[\alpha, \beta]$. Hence $\frac{\phi(t)}{\gamma(t) - z} = \frac{\phi(\alpha)}{\gamma(\alpha) - z}$, and because $\phi(\alpha) = 1$ by (3.7), we have $\phi(t) = \frac{\gamma(t) - z}{\gamma(\alpha) - z}$. Thus it follows from γ being a closed path, i.e. $\gamma(\alpha) = \gamma(\beta)$, that $\phi(\beta) = 1$.

Moreover, by Theorem 10.7. in [35] we easily see that $\mathrm{Ind}_\gamma(z) \in \mathrm{H}(\Omega)$. And since $\mathrm{Ind}_\gamma(z)$ is continous, the images of connected sets are again connected. As Ind_γ is integer-valued, we obtain that it is constant in each component. Furthermore, by (3.5) we have that $|\mathrm{Ind}_\gamma(z)| < 1$ for sufficiently large z so that $\mathrm{Ind}_\gamma = 0$ for the unbounded component of Ω. □

Remark. Note that the index number indicates how many times the path γ "winds" around a point z, i.e. the net increase of the argument of $\gamma(t) - z$ from $t = \alpha$ to $t = \beta$. This can be directly seen by looking at the imaginary part of the antiderivative of $\frac{\gamma'(s)}{\gamma(s) - z}$ which is the complex logarithm $\log|\gamma(s) - z| + \mathrm{i}\arg(\gamma(s) - z)$. While the real part cancels out, the phase difference can be integer multiples of 2π by $\arg(\gamma(\beta)) = k2\pi + \arg(\gamma(\alpha))$ with $k \in \mathbb{Z}$. The most interesting case is when $\mathrm{Ind}_\gamma = 1$, which holds for the prominent example when γ is the circle with radius r around a, i.e. $\gamma(t) = a + r\mathrm{e}^{\mathrm{i}\theta t}$ with $\theta \in [0, 2\pi]$. Then we can easily compute

$$\mathrm{Ind}_\gamma(a) = \frac{1}{\mathrm{i}2\pi}\int_\gamma \frac{\mathrm{d}z}{z-a} = \frac{r}{2\pi}\int_0^{2\pi} (r\mathrm{e}^{\mathrm{i}t})^{-1}\mathrm{e}^{\mathrm{i}t}\mathrm{d}t = 1\,.$$

Thus $\mathrm{Ind}_\gamma(z) = 1$ for $|z-a| < r$ and 0 outside.

One of the most important theorems for holomorphic functions can now be stated

Theorem 3.6 (Cauchy's integral formula)**.** *Let γ be a closed path in a convex open set Ω and $f \in \mathrm{H}(\Omega)$. If $z \in \Omega$ and $z \notin \gamma^*$, then*

$$f(z)\mathrm{Ind}_\gamma(z) = \frac{1}{\mathrm{i}2\pi}\int_\gamma \frac{f(\xi)}{\xi - z}\mathrm{d}\xi \tag{3.8}$$

In order to prove Cauchy's integral formula we need the following lemma.

Lemma 3.7. *Let Ω be a convex open set and $p \in \Omega$, f continuous on Ω and $f \in \mathrm{H}(\Omega \setminus \{p\})$. Then $f = F'$ for some $F \in \mathrm{H}(\Omega)$ and we have*

$$\int_\gamma f(z)\mathrm{d}z = 0$$

for every closed path γ in Ω.

Proof. A proof can be found in [35] Theorem 10.14. It is based on another theorem which states that if $F \in \mathrm{H}(\Omega)$ and F' continuous in Ω, then $\int_\gamma F'(z)\mathrm{d}z = 0$. □

Proof of Theorem 3.6. Let

$$g(\xi) = \begin{cases} \frac{f(\xi)-f(z)}{\xi - z} & \text{if } \xi \neq z,\ \xi \in \Omega \\ f'(z) & \text{if } \xi \neq z \end{cases} \tag{3.9}$$

which fulfills the conditions of f in Lemma 3.7 with $p = z$ so that we obtain $\frac{1}{\mathrm{i}2\pi}\int_\gamma g(\xi)\mathrm{d}\xi = 0$. The theorem follows immediately. □

Theorem 3.6 can be extended to non-convex sets

Theorem 3.8 (Cauchy's theorem)**.** *Let γ be a closed path in an arbitrary open set Ω and $f \in \mathrm{H}(\Omega)$. If $z \in \Omega$ and $z \notin \gamma^*$, then*

$$f(z)\mathrm{Ind}_\gamma(z) = \frac{1}{\mathrm{i}2\pi}\int_\gamma \frac{f(\xi)}{\xi - z}\mathrm{d}\xi \tag{3.10}$$

and

$$\int_\gamma f(z)\mathrm{d}z = 0\,. \tag{3.11}$$

Proof. See [35] Theorem 10.35. By defining $F(z) = (z - a)f(z)$ with $a \in \Omega \notin \gamma^*$ and applying (3.10) on $F(z)$ we obtain (3.11) since $F(a) = 0$. □

Using Cauchy's theorem one can readily establish the residue theorem which will be a main ingredient for the interpolation in Bernstein spaces. First of all, we need to introduce the notion of residues.

Definition 3.9 (Residue)**.** For $Q = \sum_{k=1}^{m} c_k(z - a)^{-k}$, if $f - Q$ has a removable singularity at a, i.e. the function can be extended to the Ω including a such that it is holomorphic in Ω, then c_1 is called the residue of f at a and we write $c_1 = \mathrm{Res}(f; a)$.

Theorem 3.10 (Residue theorem)**.** *Let A be the set of points at which f has poles and f be meromorphic in an open set $\Omega \supset A$, i.e. $f \in \mathrm{H}(\Omega \setminus A)$. Furthermore let γ be a path in $\Omega \setminus A$. Then we have*

$$\frac{1}{\mathrm{i}2\pi}\int_\gamma f(z)\mathrm{d}z = \sum_{a\in A} \mathrm{Res}(f; a)\mathrm{Ind}_\gamma(a)$$

Proof. Define $B = \{a \in A : \mathrm{Ind}_\gamma(a) \neq 0\} = \{a_1, \dots, a_n\}$. Let's denote by m_i the order of the poles a_i. Then by Theorem 10.21. in [35] there exist $c_{i,k}$ such that $f(z) - Q_i(z) := f(z) - \sum_{k=1}^{m_i} \frac{c_{i,k}}{(z-a_i)^k}$ has removable singularities at a_i. Note that by this definition the residues of Q_i are equal to the residues of f. Now define $g = f - \sum_i Q_i$. Then $g \in \mathrm{H}(\Omega \setminus (A \setminus B))$ has removable singularities in a_i and by Cauchy's theorem 3.8 we have that $\int_\gamma g(z)\mathrm{d}z = 0$ and thus

$$\frac{1}{\mathrm{i}2\pi}\int_\gamma f(z)\mathrm{d}z = \sum_{k=1}^{n} \frac{1}{\mathrm{i}2\pi}\int_\gamma Q_k(z)\mathrm{d}z = \sum_{k=1}^{n} \mathrm{Res}(Q_k; a_k)\mathrm{Ind}_\gamma(a_k)$$

from which the result follows. □

Remark. The theorem implies that whenever we have γ similar to a closed circle, the path integral over the function will equal to the sum of the residues at the singularities inside the circle.

3.1.3 Analytic and entire functions

In the following sections, certain types of holomorphic functions are introduced which by the extended Paley-Wiener theorem correspond to Fourier-Laplace transforms of distributions with finite support. We will however start off by noting that every holomorphic function can be locally presented by power series.

Definition 3.11 (Analytic functions)**.** A function is called analytic whenever it is locally representable by a convergent power series.

Theorem 3.12. *A function is holomorphic in Ω if and only if it is analytic in Ω, i.e. it is locally representable by power series in Ω.*

Proof. We follow the proof of [35] Theorem 10.16. Sufficiency is trivial. In order to show necessity, let us choose $\gamma_{a,r}(s) = a + r\mathrm{e}^{\mathrm{i}s}$ with radius $r < R$, $s \in [0, 2\pi]$ while the open disc $D(a; R) = \{z : |z - a| < R\}$ is a convex subset of Ω and $\gamma^* \subset D(a; R)$. Now we can apply Theorem 3.6 and obtain

$$f(z) = \frac{1}{\mathrm{i}2\pi} \int_\gamma \frac{f(\xi)}{\xi - z} \mathrm{d}\xi \,. \tag{3.12}$$

Because $\left| \frac{z-a}{\gamma_{a,r}(s)-a} \right| \leq |z - a|/r \leq$ for all $z \in D(a; r)$, we can write

$$\frac{1}{\gamma_{a,r}(s) - z} = \sum_{n=0}^{\infty} \frac{(z - a)^n}{(\gamma_{a,r}(s) - a)^{n+1}}$$

which converges uniformly on $[0, 2\pi]$. Substituting this into (3.12), by interchanging summation and integration we can conclude that there exists a sequence $\{c_n\}$ such that for all $z \in D(a; R)$

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n \tag{3.13}$$

with

$$c_n = \frac{1}{\mathrm{i}2\pi} \int_0^{2\pi} \frac{f(\gamma_{a,r}(s))\gamma'_{a,r}(s)\mathrm{d}s}{(\gamma_{a,r}(s) - a)^{n+1}} \,. \tag{3.14}$$

□

Remark. Note that by (3.14) the convergence of the series in (3.13) is only given for z inside a specific convergence radius. For arbitrary z, such a representation is always possible by choosing $r > |z|$ for the path $\gamma_{a,r}$ so that (3.13) with (3.14) still converges.

Definition 3.13 (Entire functions)**.** A function is called entire when it is holomorphic in the entire complex plane.

3.1.4 Bernstein and Paley-Wiener spaces

Next we define special types entire functions. First we have to introduce two notations: If $h(r) < \phi(r)$ holds whenever r is large enough we write $h(r) \overset{\text{as}}{<} \phi(r)$. If $h(r) < \phi(r)$ holds for some sequence $r_n \to \infty$, we shall write $h(r) \overset{\text{n}}{<} \phi(r)$. We also define

$$M_f(r) = \sup_{|z|=r} |f(z)| \,.$$

Definition 3.14 (Finite order)**.** An entire function is said to be of finite order ρ_o if ρ_o is the greatest lower bound of those values of ρ for which the following inequality is fulfilled

$$M_f(r) \overset{\text{as}}{<} \mathrm{e}^{r^\rho} \,. \tag{3.15}$$

Definition 3.15 (Exponential type)**.** If for an entire function we have

$$M_f(r) \overset{\text{as}}{<} \mathrm{e}^{\sigma r} \tag{3.16}$$

with σ_T being the greatest lower bound those values of σ for which the inequality is fulfilled, then we call f a function of exponential type σ_T. It follows that

$$\mathrm{e}^{(\sigma_T-\epsilon)r} \overset{n}{<} M_f(r) \overset{\text{as}}{<} \mathrm{e}^{(\sigma_T+\epsilon)r}$$

for all $\epsilon > 0$.

In the following $p \in [1, \infty]$.

Definition 3.16 (Paley-Wiener space)**.** An entire function of exponential type σ is an element of the Paley-Wiener space $\mathcal{PW}_\sigma^p$ if it is representable by

$$f(z) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} g(\omega)\mathrm{e}^{\mathrm{i}\omega z}\mathrm{d}\omega \tag{3.17}$$

with $g \in \mathcal{L}^p([-\sigma, \sigma])$. The norm of $f \in \mathcal{PW}_\sigma^p$ is defined as the $\mathcal{L}^p([-\sigma, \sigma])$ norm of g, i.e.

$$\|f\|_{\mathcal{PW}_\sigma^p} = \int_{-\sigma}^{\sigma} |g(x)|^p \mathrm{d}x \tag{3.18}$$

Remark. Note that it is important here that g is in the Lebesgue space. By the Paley-Wiener Theorem 2.23 we know that whenever we have an entire function of exponential type σ, there exists a distribution with support $[-\sigma, \sigma]$ and vice versa. Paley-Wiener spaces only include entire functions which have functions in $\mathcal{L}^p$ as their Inverse Fourier Transform.

Lemma 3.17. *For* $1 \le p < s \le \infty$ *we have* $\mathcal{PW}_\sigma^s \subset \mathcal{PW}_\sigma^p \subset \mathcal{PW}_\sigma^1$

Proof. Without loss of generality, let ζ be the only singularity of $g \in \mathcal{L}^p(H)$ with compact support $H = [-\sigma, \sigma]$. We can split the integral in (3.18) as follows

$$\begin{aligned} \int_{-\sigma}^{\sigma} |g(x)|^p \mathrm{d}x = \int_{-\sigma}^{\zeta-\epsilon} |g(x)|^p \mathrm{d}x + \int_{\zeta+\epsilon}^{\sigma} |g(x)|^p \mathrm{d}x \\ + \int_{\zeta-\epsilon}^{\zeta+\epsilon} |g(x)|^p \mathrm{d}x \end{aligned} \tag{3.19}$$

with $\epsilon > 0$. A problem can only originate from the third term. If we choose ϵ small enough such that for all $|x-\zeta| < \epsilon$ small enough $|g(x)| > 1$, we have $|g(x)|^p < |g(x)|^s$ if $p < s < \infty$. The subsequent implication easily follows

$$\lim_{\epsilon\to 0} \int_{\zeta-\epsilon}^{\zeta+\epsilon} |g(x)|^s \mathrm{d}x < \infty \implies \lim_{\epsilon\to 0} \int_{\zeta-\epsilon}^{\zeta+\epsilon} |g(x)|^p \mathrm{d}x < \infty$$

and thus $\int_H |g(x)|^s \mathrm{d}x < \infty \implies \int_H |g(x)|^p \mathrm{d}x$. This shows that $\mathcal{L}^s(H) \subset \mathcal{L}^p(H)$ so that each $f \in \mathcal{PW}_\sigma^s$ is representable by a $g \in \mathcal{L}^p(H)$, i.e. $f \in \mathcal{PW}_\sigma^p$ and the lemma is proved. □

Definition 3.18 (Bernstein spaces)**.** An entire function of exponential type σ is said to be in the Bernstein space $\mathcal{B}_\sigma^p$ if its restriction to the real axis belongs to $\mathcal{L}^p(\mathbb{R})$. The norm of $f \in \mathcal{B}_\sigma^p$ is defined by the $\mathcal{L}^p(\mathbb{R})$ norm

$$\|f\|_{\mathcal{B}_\sigma^p} := \left(\int_{\mathbb{R}} |f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} \tag{3.20}$$

Lemma 3.19. *For* $1 \leq p < s \leq \infty$ *we have* $\mathcal{B}_\sigma^p \subset \mathcal{B}_\sigma^s \subset \mathcal{B}_\sigma^\infty$

Proof. By the Plancherel Pólya theorem which is introduced later in section 3.38 we obtain the inequality $|f(z)| \leq \frac{2}{\pi} \mathrm{e}^{\sigma|\operatorname{Im} z|}\|f\|_{\mathcal{L}^p(\mathbb{R})}$. This tells us that for entire functions f of exponential type σ, we have that $f\,|_{\mathbb{R}} \in \mathcal{L}^p(\mathbb{R})$ implies $f\,|_{\mathbb{R}} \in \mathcal{L}^\infty(\mathbb{R})$ and thus the inclusion in $\mathcal{B}_\sigma^\infty$ immediately follows. Similar as in the case of Paley-Wiener Spaces, let us split the integral in (3.20) as follows

$$\begin{aligned}\int_{\mathbb{R}} |g(x)|^p \mathrm{d}x = \int_{-\infty}^{-a} |g(x)|^p \mathrm{d}x + \int_a^\infty |g(x)|^p \mathrm{d}x \\ + \int_{-a}^{a} |g(x)|^p \mathrm{d}x\,.\end{aligned} \tag{3.21}$$

Now we can basically use the opposite argumentation as before: Since the function is bounded, integrability only depends on its behavior at infinity, i.e. the two first tems. As it is a necessary condition for $f \in \mathcal{L}^p(\mathbb{R})$ with $p < \infty$ to vanish at infinity, it is possible choose a such that $|g(x)| < 1$ for all $|x| > a$. Then we have $|f(x)|^s < |f(x)|^p$ for $p < s < \infty$ and

$$\int_a^\infty |f(x)|^p \mathrm{d}x < \infty \implies \int_a^\infty |f(x)|^s \mathrm{d}x < \infty\,.$$

Hence, since $\mathcal{L}^\infty(\mathbb{R}) \cap \mathcal{L}^p(\mathbb{R}) \subset \mathcal{L}^\infty(\mathbb{R}) \cap \mathcal{L}^s(\mathbb{R})$, we have $\mathcal{B}_\sigma^p \subset \mathcal{B}_\sigma^s$. ☐

Remark. Note that although $\mathcal{B}_\sigma^p$ are generally different from $\mathcal{PW}_\sigma^p$ and no particular inclusion statements can be made other than $\mathcal{PW}_\sigma^1 \subset \mathcal{B}_\sigma^\infty$, by Theorem 2.24 we have for $p = 2$ that $\mathcal{B}_\sigma^2 = \mathcal{PW}_\sigma^2$.

3.1.5 Sine-type functions

Definition 3.20 (Sine-type functions)**.** An entire function $f(z)$ of exponential type σ is called sine-type function of type σ if
(i) the zeros $\{\lambda_k\}$ of $f(z)$ are separated, i.e. $\inf_{k\neq n} |\lambda_k - \lambda_n| = 2\delta > 0$ and
(ii)

$$\sup_k |\operatorname{Im} \lambda_k| = H < \infty \tag{3.22}$$

(iii) and there exist constants h, c, C such that

$$0 < c < |f(x + \mathrm{i}h)| < C < \infty, \; -\infty < x < \infty$$

Remark. From the definition we see right away that the zeros of a sine-type function lie in a horizontal strip parallel to the real axis.

An alternative definition is often used and established in the subsequent lemma.

Lemma 3.21. *If f is a sine-type function of type σ, then (i) the zeros $\{\lambda_k\}$ of $f(z)$ are separated, i.e.* $\inf_{k\neq n}|\lambda_k-\lambda_n| = 2\delta > 0$ *and*
(ii) there exist positive constants A, B, H such that

$$Ae^{\sigma|y|} \leq |f(x+iy)| \leq Be^{\sigma|y|} \tag{3.23}$$

for real x, y and whenever $|y| \geq H$.

Sketch of proof. The lemma follows from the fact that given Definition 3.20, for any $\eta > 0$ there exists $m_\eta > 0$ such that

$$|f(z)| > m_\eta e^{\sigma|\operatorname{Im} z|}$$

if $\operatorname{dist}(z, \{\lambda_k\}) > \eta$. It is intuitive to see that outside of circles around λ_k the function is non-zero. Levin gives a rigorous proof for this statement in [22] Lecture 22 Lemma 1. To obtain the concrete estimate he further uses the Nevanlinna representation, more precisely Corollary 2 to Theorem 1 in Lecture 14.

Lemma 3.22. *For every sine-type function f there exist constants N_1, N_2 such that*

$$0 < N_1 < |f'(\lambda_k)| < N_2 < \infty$$

Sketch of proof. The proof uses (3.23) and the maximum and minimum principles for analytic functions.

There are a lot of nice theorems which show that minimally shifted zeros of sine-type functions are still zeros of sine-type functions, such as the Katsnelsons Theorem or the 1/4 Theorem (see [22] Lecture 23.2). The next theorem also belongs to this type of result.

Theorem 3.23 (Levin)**.** *Let a function be representable by* $S(z) = \lim_{n\to\infty}\prod_{|k|<n}\left(1-\frac{z}{\lambda_k}\right)$ *with zeros $\{\lambda_k\}$ which have separated real parts. A necessary and sufficient condition for $S(z)$ to be of sine-type is that*

$$S_R(z) = \lim_{n\to\infty}\prod_{|k|<n}\left(1-\frac{z}{\operatorname{Re}\lambda_k}\right) \tag{3.24}$$

is also a function of sine type.

Proof. An easy proof can be found in [23] for Lemma 1. □

Remark. An immediate consequence is that by shifting the imaginary parts of the zeros of a sine-type function by an arbitrary finite amount, we still have zeros a sine-type function.

3.2 Fundamental principles

In this section the aim is to derive the Plancherel Pólya Theorem upon which the convergence of the interpolation formulas in the next chapter is are based. It gives us an estimate of the form

$$\int_{\mathbb{R}} |f(x+\mathrm{i}y)|^p \mathrm{d}x \leq \mathrm{e}^{p\sigma|y|} \|f\|_{\mathcal{B}_\sigma^p} . \tag{3.25}$$

3.2.1 Phragmen Lindelöf theorems

Some of the most important principles used to study the growth behavior of holomorphic or entire functions are the Phragmen-Lindelöf theorem and its corollaries. It is based on the maximum modulus principle,which will thus be introduced first. This section closely follows [22] lecture 6. Note that when we write that a function is bounded on X by a constant M this constant is chosen as the smallest upper bound, i.e. $M = \sup_{z\in X} |f(z)|$.

Theorem 3.24 (Maximum modulus)**.** *Let $f(z)$ be a holomorphic function in a region Ω and continuous on the closed set $\overline{\Omega}$. If $|f(z)| \leq M$ on the boundary $\partial\Omega$, then $|f(z)| < M$ at all interior points of Ω unless $f(z)$ is a constant such that $|f(z)| = M$ everywhere.*

Proof. The proof using the power series representation of f is taken from [39]. Let $z_0 \in \Omega$ be arbitrary and $f \in \mathrm{H}(\Omega)$. Hence by Theorem 3.12, we can write $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ inside a certain disc $D(z_0; R)$. It is easy to see that

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + r\mathrm{e}^{\mathrm{i}\theta})|^2 \mathrm{d}\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

for $r < R$. Now assume that $|f|$ attains a local maximum at an interior point $z_0 \in \Omega$, i.e. $|f(z_0 + r\mathrm{e}^{\mathrm{i}\theta})| \leq |f(z_0)| \,\forall\theta$ for $D(z_0; r) \subset \Omega$. Then, the left-hand side does not exceed $|f(z_0)|^2 = |a_0|^2$. Hence

$$|a_0|^2 + |a_1|^2 r^2 + |a_2|^2 r^4 + \cdots \leq |a_0|^2$$

for a positive value of r. Therefore $|a_i| = 0\,\forall i \neq 0$ and $|f(z)|$ is a constant which contradicts the assumption. Thus we can conclude that $|f(z)| < M$ for every interior point $z \in \Omega$ unless $|f|$ is a constant in the interior of Ω. Furthermore if $|f|$ is constant in the interior of Ω and $|f(z)| \leq M$ on $\partial\Omega$, then by continuity on $\overline{\Omega}$, $|f(z)| = M$ for all $z \in \Omega$ and the proof is complete. □

Next we will consider the Phragmen Lindelöf theorems which provide different growth conditions for analytic functions inside angles $D = \{z : \alpha < \arg z < \beta\}$ and we set

$$M_f(r) = \sup\{|f(r\mathrm{e}^{\mathrm{i}\theta})| : \alpha < \theta < \beta\} .$$

Theorem 3.25 (Phragmen-Lindelöf 1)**.** *Let D be an angle of opening π/λ and let $f(z)$ be a function analytic in D satisfying the asymptotic estimate*

$$M_f(r) \overset{as}{<} \mathrm{e}^{r^\rho} \tag{3.26}$$

with $\rho < \lambda$. If $f(z)$ is bounded by a constant M on the sides of D, then $|f(z)| \leq M$ for all $z \in \mathcal{D}$.

Proof. First we assume without loss of generality that the angle is symmetric around the real axis $D = \{r\mathrm{e}^{\mathrm{i}\theta} : |\theta| < \alpha\}$ with $\alpha = \pi/2\lambda$. Now it is possible to choose a number ρ_1 such that $\rho < \rho_1 < \lambda$ and set

$$\phi_\delta(z) = f(z)\mathrm{e}^{-\delta z^{\rho_1}} \; \forall \delta > 0 \, .$$

ϕ is holomorphic in D and continuous in $\overline{D}$. By simply taking the modulus and using the finite order property of f, we obtain

$$|\phi_\delta(z)| \overset{\text{as}}{<} \mathrm{e}^{|z|^\rho - \delta|z|^{\rho_1}\cos(\rho_1\alpha)}$$

inside the whole angle D. Since $\rho < \rho_1$ and $\cos(\rho_1\alpha) > 0$, the exponent becomes negative for large enough $|z|$, so that there exists a R_δ such that for all $R > R_\delta$ the following inequality holds

$$|\phi_\delta(R\mathrm{e}^{\mathrm{i}\theta})| \leq M, \;\; -\alpha < \theta < \alpha \, .$$

Now applying Theorem 3.24 to the function $\phi_\delta(z)$ we find that $|\phi_\delta(z)| \leq M$ inside the sector $D = \{r\mathrm{e}^{\mathrm{i}\theta} : r < R, \, |\theta| < \alpha\}$, i.e.

$$|f(z)| \leq M\mathrm{e}^{\delta|z|^{\rho_1}} \; \forall z \in D \, . \tag{3.27}$$

When $R \to \infty$ we see that the inequality holds everywhere inside the angle D and especially also for arbitrarily small δ which completes the proof. □

Remark. From this very powerful theorem, many other corollaries can be deduced. First of all it can be noted as an immediate consequence, that an entire function of order $\rho < 1$ which is bounded on a line, i.e. $\lambda = 1$, must reduce to a constant. Assuming that the line is the imaginary axis, i.e. $\sup_{y\in\mathbb{R}} |f(iy)| = M$, this can be seen by applying Theorem 3.25 twice on the positive and negative real half plane. If the interior had values $|f(z)| \leq \tilde{M} < M$ at particular angles α, β then one could use the Phragmen Lindelöf Theorem once more and deduce that $|f(iy)| \leq \tilde{M} < M$ which contradicts the assumption.

For entire functions of exponential type $\rho = 1$, Theorem 3.25 is restricted to angles of openings strictly smaller than π. The next corollary includes the case $|\theta| = \pi$ for such functions, but instead of an absolute bound, it yields a growth condition.

Corollary 3.26 (Phragmen-Lindelöf 2)**.** *If a function $f(z)$ analytic inside an angle $D = \{z : |\arg z| < \alpha = \frac{\pi}{2\rho}\}$ satisfies*

$$M_f(r) \overset{as}{<} \mathrm{e}^{(\sigma+\epsilon)r^\rho} \tag{3.28}$$

for all $\epsilon > 0$ and if $f(z)$ is bounded on the sides of D by a constant M, then

$$|f(r\mathrm{e}^{\mathrm{i}\theta})| \leq M\mathrm{e}^{\sigma r^\rho \cos(\rho\theta)}, \;\; r\mathrm{e}^{\mathrm{i}\theta} \in D\,. \tag{3.29}$$

Proof. In the following, we will call the sets $R_\theta = \{z = r\mathrm{e}^{\mathrm{i}\theta} : r \geq 0\}$ rays with angle θ. Define

$$\phi_\epsilon(z) = f(z)\mathrm{e}^{-(\sigma+\epsilon)z^\rho}\,. \tag{3.30}$$

Given the assumption (3.28), it follows that for any ϵ we can choose $\epsilon' < \epsilon$ such that for r large enough, we have

$$|\phi_\epsilon(r\mathrm{e}^{\mathrm{i}\theta})| \leq \mathrm{e}^{(\sigma+\epsilon')r^\rho - (\sigma+\epsilon)r^\rho\cos(\rho\theta)} \tag{3.31}$$

Since f is bounded by M on $R_{\pm\alpha}$, we obtain for ϕ_ϵ:

$$\begin{aligned}\sup_{r\geq 0}\left|\phi_\epsilon(r\mathrm{e}^{\pm\mathrm{i}\alpha}\right| &= \sup_{r\geq 0}\left|f(r\mathrm{e}^{\pm\mathrm{i}\alpha})\right|\mathrm{e}^{-(\sigma+\epsilon)r^\rho\cos(\rho\alpha)}\\ &= M\end{aligned}$$

since $\cos(\rho\alpha) > 0$.

We can then find θ_0 such that

$$u(\sigma,\epsilon,\epsilon',\theta) = (\sigma+\epsilon') - (\sigma+\epsilon)\cos(\rho\theta) < 0 \tag{3.32}$$

and thus $|\phi_\epsilon(r\mathrm{e}^{\mathrm{i}\theta})| \to 0$ as $r \to \infty$ for all $|\theta| \leq \theta_0$ (see Fig. 3.1). These rays are then bounded by M_1. Pick for example $\theta_0 = 0$ and assume that ϕ is not constant in D and $M_1 > M$ is the smallest upper bound for ϕ_ϵ on this ray. We will quickly show that this cannot hold. When we choose $\lambda = 2\rho$ and $\rho = \rho$, the assumptions for Theorem 3.25 hold, since $|\phi_\epsilon(r\mathrm{e}^{\mathrm{i}\theta})| \leq \mathrm{e}^{ur^\rho} \overset{\text{as}}{<} \mathrm{e}^{r^{\rho'}}$ for $\theta \in (-\alpha,\alpha)$ and any $\rho < rho' < lambda$. Now by Theorem 3.25 we have for region $D_1 = \{z : 0 < \arg z < \alpha = \frac{\pi}{4\rho}\}$ that $|\phi_\epsilon(z)| < \max(M, M_1) = M_1$. Since $M_1 > M$, we have $\sup_r |\phi_\epsilon(r\mathrm{e}^{\mathrm{i}\theta_1})| = N_1 < M_1$ for some ray in D_1, corresponding to an angle θ_1. The same can be said of the region $D_2 = \{z : -\frac{\pi}{4\rho} < \arg z < 0\}$ where we pick a ray on which $\sup_r |\phi_\epsilon(r\mathrm{e}^{\mathrm{i}\theta_2})| = N_2 < M_1$ (see Fig. 3.2). Now define $\frac{\pi}{\lambda'} = \theta_1 + \theta_2 < \frac{\pi}{2\rho}$ such that we have $\lambda' > 2\rho$. The Phragmen Lindelöf Theorem can again be used to obtain for the real axis with $\theta_0 = 0$ that

$$|\phi_\epsilon(r)| \leq \max(N_1, N_2) < M_1 \qquad \forall r \in \mathbb{R}$$

This contradicts the original assumption that M_1 is the lowest upper bound so that we have proven $M_1 \leq M$.

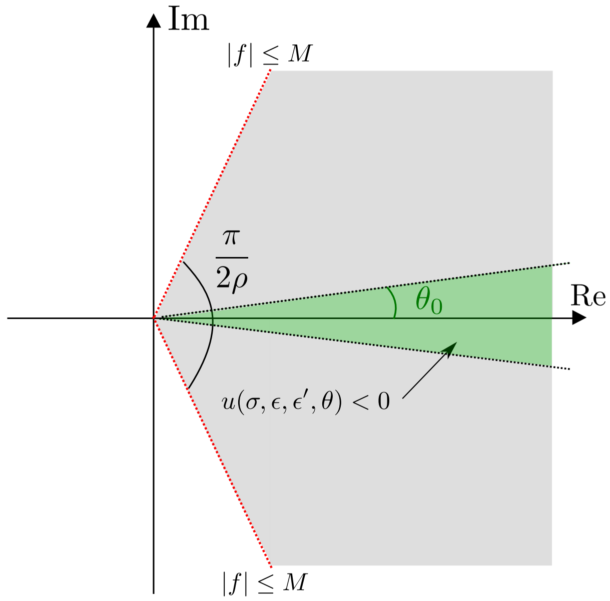


Figure 3.1: This illustration depicts the range of θ for which the exponent $u(\sigma, \epsilon, \epsilon', \theta) < 0$ in (3.32) and $f(z)$ is bounded in each ray with $|\theta| < \theta_0$.

Now we can again use Theorem 3.25 for $D_+ = \{z : -\delta < \arg z < \frac{\pi}{2\rho}\}$ and $D_- = \{z : \frac{\pi}{2\rho} < \arg z < \delta\}$ separately with arbitrary $\delta > 0$ such that $D = D_+ \cup D_-$ and $u < 0$. The above discussion shows that $\sup_r |\phi_\epsilon(r\mathrm{e}^{\mathrm{i}\delta})| \leq M$ for all feasible δ so that $|\phi_\epsilon(z)| \leq M$ for all $z \in D$. Then we have that $|\phi_\epsilon(z)| \leq M \; \forall \delta > 0 \, , z \in D$. Now letting $\epsilon \to 0$ yields the corollary. □

The most important case for us is when $\rho = 1$, i.e. the functions are of exponential type. For these functions we can deduce a rather simple but useful corollary.

Corollary 3.27 (Phragmen-Lindelöf 3)**.** *If $f(z)$, $z = x + \mathrm{i}y$ is an analytic function in the half-plane $I_+ := \{z : \operatorname{Im} z > 0\}$ such that for all $\epsilon > 0$*

$$M_f(r) \overset{as}{<} \mathrm{e}^{(\sigma+\epsilon)r}$$

and $|f(x)| \leq M$ on the real axis, then

$$|f(x + \mathrm{i}y)| \leq M\mathrm{e}^{\sigma y} \, .$$

Proof. We are interested in $f(z)$ on the half-plane I_+, which is equivalent to looking at the behavior of $g(z) := f(\mathrm{i}z)$ in the positive real half-plane $R_+ := \{z : \operatorname{Re} z > 0\}$ (see Fig. 3.3). The given assumptions that $|f(x)| \leq M$ for all $x \in \mathbb{R}$ and $f(z)$ analytic

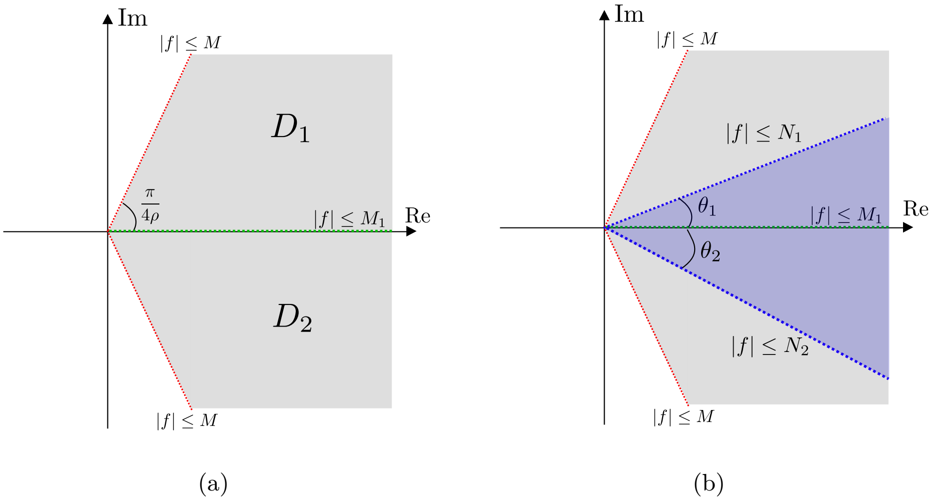


Figure 3.2: (a) shows the two regions on which the Phragmen Lindelöf Theorem is applied to obtain rays at angles θ_1 and θ_2 in (b) on which the function is bounded by a constant N_1 and N_2 respectively. If we assume $M_1 > M$ then $N_1 < M_1$ and $N_2 < M_1$ since the function is not constant. But then using Phragmen Lindelöf again on the blue angle would yield at $\theta_0 = 0$ that $|f| < M_1$ which contadicts the assumption $|f| \leq M_1$ with M_1 being the smallest upper bound. Thus $M_1 \leq M$.

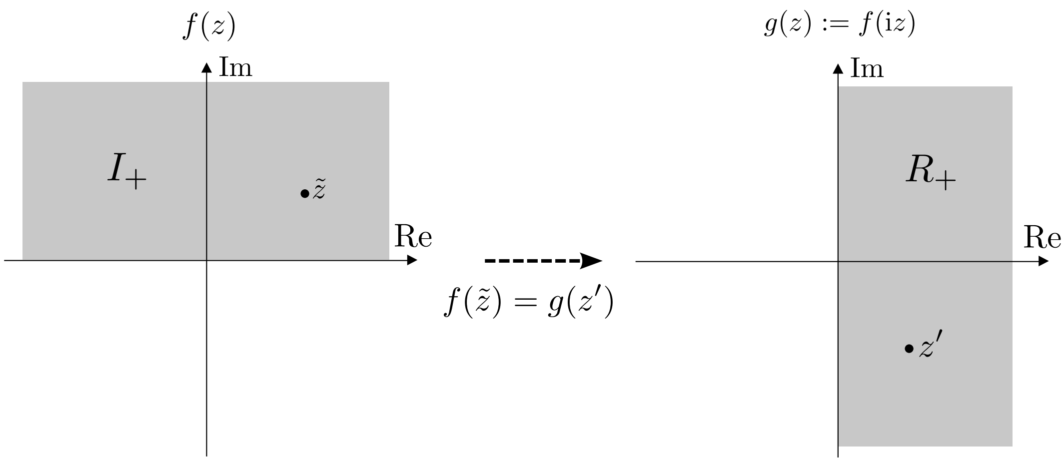


Figure 3.3: Analysis of the behavior of $f(z)$ in the region I_+ is equivalent to the analysis of $g(z) := f(\mathrm{i}z)$ in R_+.

in I_+ now translate to the function $g(z)$ as $|g(\mathrm{i}y)| \leq M$ and g analytic on R_+. One can then use Corollary 3.26 and obtain with $z' = x' + \mathrm{i}y'$ and $\tilde{z} = \tilde{x} + \mathrm{i}\tilde{y}$

$$|g(z')| = |f(\mathrm{i}z')| \leq M\mathrm{e}^{\sigma x'} \;\; \forall z' \in R_+$$

from which it follows that

$$|f(\tilde{z})| \leq M\mathrm{e}^{\sigma\tilde{y}} \;\; \forall \tilde{z} \in I_+ \, .$$

□

Remark. For entire functions which are analytic on both half-planes we obtain the more general general result

$$|f(x + \mathrm{i}y)| \leq M\mathrm{e}^{\sigma|y|} \, . \tag{3.33}$$

3.2.2 Subharmonic functions

Let us recall that the final goal was to establish the Plancherel Pólya Theorem. In this theorem we are dealing with integrals of the form $\int_{\mathbb{R}} |f(x)|^p \mathrm{d}x$ so that it is necessary to introduce subharmonic functions for which modified versions of the Phragmen Lindelöf Theorems above also hold.

Definition 3.28 (Subharmonic)**.** A real function $u(z) < \infty$ is called subharmonic in a domain D if at each point $z_0 \in D$ it satisfies two conditions:
(i) upper semicontinuity, i.e.

$$u(z_0) = \lim_{\delta \to 0} \sup_{|z - z_0| < \delta} u(z) \tag{3.34}$$

(ii) and the mean-value property

$$u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + r\mathrm{e}^{\mathrm{i}\theta}) \mathrm{d}\theta \tag{3.35}$$

for small enough r.

The following statements establish the relationship between holomorphic and subharmonic functions, which are essential for the proof of the Plancherel Pólya Theorem.

Theorem 3.29. *If $f \in \mathrm{H}(\Omega)$ and $f \not\equiv 0$, i.e. if f is nonzero and holomorphic in a region Ω, then $\log|f|$ is subharmonic in Ω.*

The proof of this theorem requires the following two lemmata.

Lemma 3.30 (Jensen's formula)**.** *Suppose* $\Omega = D(0;R)$*,* $f \in \mathrm{H}(\Omega)$*,* $f(0) \neq 0$ *and for* $0 < r < R$*,* $z_1, \ldots, z_N$ *are the zeros of* f *in* $\bar{D}(0;r)$ *listed according to their multiplicities. Then*

$$|f(0)| \prod_{n=1}^{N} \frac{r}{|z_n|} = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(r\mathrm{e}^{\mathrm{i}\theta})|\mathrm{d}\theta\right). \tag{3.36}$$

Proof. See Theorem 15.18. in [35]. □

Let's denote by $\mathcal{H}^\infty$ the space of bounded holomorphic functions not identically 0 on the open unit disc $D(0,1)$ with norm $\|f\|_{\mathcal{H}^\infty} = \sup_{z\in D(0,1)} |f(z)|$. Since we can use the same arguments for scaled and translated discs, we assume without loss of generality that for each $f \in \mathrm{H}(\Omega)$ we have $f \in \mathcal{H}^\infty$ so that the following lemma can be used to prove the theorem.

Lemma 3.31. *For* $f \in \mathcal{H}^\infty$ *we can define*

$$\mu_r(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(r\mathrm{e}^{\mathrm{i}\theta})|\mathrm{d}\theta \ \ (0 < r < 1)$$
$$\mu^*(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f^*(\mathrm{e}^{\mathrm{i}\theta})|\mathrm{d}\theta$$

where $f^*(\mathrm{e}^{\mathrm{i}\theta}) = \lim_{r\to 1} f(r\mathrm{e}^{\mathrm{i}\theta})$*. Then we have*

$$\mu_r(f) \leq \mu_s(f) \ \ \textit{if}\ 0 < r < s < 1 \tag{3.37}$$
$$\mu_r(f) \to \log|f(0)| \ \ \textit{as}\ r \to 0 \tag{3.38}$$

Proof. In order to be able to apply Jensen's formula we need to find $g(z)$ such that $g(0) \neq 0$ while still $g \in \mathcal{H}^\infty$. This can be achieved by a certain m with $g(z) = z^{-m} f(z)$ (see [35] Theorem 10.18). From Jensen's formula we immediately see that $\mu_r(g) \leq \mu_s(g)$ for $r < s$ because the left hand side of (3.36) cannot decrease when r increases. Since also $\mu_r(f) = \mu_r(g) + m\log(r)$, (3.37) holds. Now in order to see (3.38) we choose $|f| \leq 1$ without loss of generality and first observe that $\lim_{r\to 0} f(r\mathrm{e}^{\mathrm{i}\theta}) = f(0)$. Then using Fatou's lemma on exchangeability of the integral and the limit we get

$$\lim_{r\to 0} \mu_r(f) \geq \frac{1}{2\pi}\int_{-\pi}^{\pi} \lim_{r\to 0} \log|f(r\mathrm{e}^{\mathrm{i}\theta})| = \log|f(0)|. \tag{3.39}$$

Furthermore, from

$$\lim_{r\to 0}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(r\mathrm{e}^{\mathrm{i}\theta})| + \log\frac{1}{|f(r\mathrm{e}^{\mathrm{i}\theta})|}\right]\mathrm{d}\theta = 0$$

and using Fatou's lemma again, it follows that

$$\lim_{r\to 0} \mu_r(f) = \lim_{r\to 0} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(r\mathrm{e}^{\mathrm{i}\theta})| \leq -\frac{1}{2\pi}\int_{-\pi}^{\pi} \lim_{r\to 0} \frac{1}{\log|f(r\mathrm{e}^{\mathrm{i}\theta})|} = \log|f(0)| \tag{3.40}$$

Combining (3.39) and (3.40) yields (3.38). □

Proof of Theorem 3.29. We reproduce the proof for [35] Theorem 17.3.
Direct application of (**??**) and (**??**) in Lemma 3.31 first yields that $\log|f(0)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(re^{i\theta})d\theta$ for $0 < r < 1$. As mentioned before, the results can be extended to arbitrary translated and scaled discs $D(a;r)$ from which it follows that $\log|f|$ fulfills the mean-value property. Upper semicontinuity now follows from continuity of $|f(z)|$ and $\log u$ for $u \neq 0$ and because $\log|f(z)| = -\infty$ for $f(z) = 0$. □

Corollary 3.32. *Under the same conditions as in Theorem 3.29, $|f|^p$ is also subharmonic in Ω.*

Proof. We use the well-known fact that if ϕ is a monotonically increasing convex function on $\mathbb{R}$ and u is subharmonic on Ω, then the composition $\phi \circ u$ is also subharmonic (see [35] Theorem 17.2.). Since $\phi(u) = e^u$ is monotonically increasing and convex and $u(z) = \log|f(z)|$ is subharmonic, the corollary follows. □

Lemma 3.33. *If f is subharmonic in Ω, then*

$$G_a(z) = \int_{-a}^{a} f(z+t)dt$$

is also subharmonic in Ω for all $a \in \mathbb{R}$.

Proof. Let us first prove the mean-value property (3.35). We have

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} G_a(z_0 + re^{i\theta})d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-a}^{a} f(z_0 + t + re^{i\theta})dt d\theta \\
&= \int_{-a}^{a}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(z_0 + t + re^{i\theta})d\theta dt \\
&\geq \int_{-a}^{a} f(z_0 + t)dt = G_a(z_0)
\end{aligned}$$

where the second equality follows from Fubini's Theorem. It is applicable because $\int_{-\pi}^{\pi}\int_{-a}^{a}|f(z_0+t+re^{i\theta})|d\theta dt < \infty$ as we integrate a continuous function over compact sets. Upper semicontinuity (3.34) of G_a directly follows from the continuity of the integral and from f being subharmonic. □

3.2.3 The Plancherel Pólya theorem

We can now establish the maximum modulus principle and Phragmen Lindelöf theorems for subharmonic functions.

Theorem 3.34 (Maximum modulus principle for subharmonic functions)**.** *If a subharmonic function $u(z)$ in a domain D attains its maximum value at an interior point $z_0 \in D$, then $u(z) \equiv$ const. Or in other words: The inequality*

$$u(z) \leq \sup_{z\in\partial D}|u(z)| \ \forall z \in D$$

is valid everywhere in D and equality only holds if $u(z)$ is constant in D.

Proof. Define $M = \sup_{z\in D} u(z)$ and assume $u(z_0) = M$ with $z_0 \in D$. By the mean-value property of subharmonic functions we have

$$u(z_0) = M \leq \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + r\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta \leq M$$

for all $r < r_0$ with r_0 small enough. Consequently, $u(z) = M$ for all $z \in D(z_0, r) = \{z : |z - z_0| < r\}$. We can now cover the interior of D by open balls where we have $u(z) = M$ everywhere. Because of upper semicontinuity we can then conclude that $u(z) = M$ for all $z \in D$. □

Theorem 3.35 (Phragmen Lindelöf for subharmonic functions)**.** *Let D be an angle of opening π/λ and let $u(z)$ be a function subharmonic in this angle satisfying an asymptotic estimate*

$$u(z) \overset{as}{<} |z|^\rho, \ \forall \rho < \lambda$$

and bounded by a constant M on the boundary of the angle. Then $|u(z)| \leq M$ inside the full angle D.

Proof. The proof follows similar arguments as in Theorem 3.25 now using the maximum modulus principle for subharmonic functions, Theorem **??**. □

The following corollaries can be proven with the same ideas as in the case of holomorphic functions.

Corollary 3.36. *Let u(z) be subharmonic in $D = \{z : |\arg z| < \alpha = \frac{\pi}{2\rho}\}$ satisfy the asymptotic inequality*

$$u(z) \overset{as}{<} (\sigma + \epsilon)|z|^\rho \ \forall \epsilon > 0\,.$$

If u(z) is bounded on the sides of D by M, then

$$u(r\mathrm{e}^{\mathrm{i}\theta}) \leq M + \sigma r^\rho \cos(\rho\theta)\,.$$

Corollary 3.37. *Let u(z) be subharmonic in I_+ and satisfy the asymptotic inequality*

$$u(z) \overset{as}{<} (\sigma + \epsilon)|z| \ \forall \epsilon > 0$$

and $u(z) \leq M$ on the real axis, then

$$u(x + \mathrm{i}y) \leq M + \sigma y$$

Remark. For subharmonic functions in the entire plane we have

$$u(x + \mathrm{i}y) \leq M + \sigma|y| \tag{3.41}$$

With these results we can finally state the Plancherel Pólya theorem.

Theorem 3.38 (Plancherel Pólya)**.** *If $f(z)$ is an entire function of exponential type σ and if for some positive number p*

$$\int_{-\infty}^{\infty} |f(x)|^p \mathrm{d}x = M < \infty\,, \tag{3.42}$$

then

$$\int_{-\infty}^{\infty} |f(x+\mathrm{i}y)|^p \mathrm{d}x \leq \mathrm{e}^{p\sigma|y|} \int_{-\infty}^{\infty} |f(x)|^p \mathrm{d}x\,. \tag{3.43}$$

Proof. In the first part of the proof we follow Young [43] Theorem 16 in section 2.2.3. and in the second part we adopt the approach by Levin [22] Lecture 7 Theorem 4. Define

$$g(z) = f(z)\mathrm{e}^{\mathrm{i}(\sigma+\epsilon)z}$$

with $|f(z)| \overset{\text{as}}{<} \mathrm{e}^{(\sigma+\epsilon)|z|}$ for all $\epsilon > 0$ and

$$G_a(z) = \int_{-a}^{a} |g(z+t)|^p \mathrm{d}t\,.$$

Then $g(z)$ is of exponential type on the half-plane I_+ as in Corollary 3.27 and $\lim_{y\to\infty} g(x+\mathrm{i}y) = 0$ uniformly in $-a \leq x \leq a$ so that $\sup_{y>0} G_a(\mathrm{i}y) = N_a < \infty$. Also, $G_a(z)$ is subharmonic by Corollary 3.32 and Lemma 3.33. Moreover by assumption (3.42) $\sup_{x\in\mathbb{R}} G_a(x) = M_a < \infty$. Then we can deduce that

$$G_a(z) \leq \max(M_a, N_a) = M_a \tag{3.44}$$

for $\operatorname{Im} z \geq 0$. The last equality is easily seen using the same argumentation as in the proof of Corollary 3.26. For the first inequality, we can find by straight-forward calculation that

$$\begin{aligned} G_a(z) &\overset{\text{as}}{<} \int_{-a}^{a} \mathrm{e}^{p(\sigma+\epsilon)(|z|+|t|)}\mathrm{e}^{-p(\sigma+\epsilon)y}\mathrm{d}t \\ &\leq C\mathrm{e}^{p(\sigma+\epsilon)|z|} \int_{-a}^{a} \mathrm{e}^{p(\sigma+\epsilon)|t|}\mathrm{d}t\,. \end{aligned}$$

Therefore, as $\log G_a$ is subharmonic by Theorem 3.29, so that we can apply Theorem 3.35 to $\log G_a$ on the angles of opening $\pi/2$ and the first inequality in (3.44) follows. Define

$$w_a(z) = \int_{-a}^{a} |f(z+t)|^p \mathrm{d}t = G_a(z)\mathrm{e}^{p(\sigma+\epsilon)y} \overset{\text{as}}{<} \mathrm{e}^{p(\sigma+\epsilon)|z|} \tag{3.45}$$

By Theorem 3.29 we know that $\log w_a(z)$ is also a subharmonic function in the entire complex plane so that by using (3.42), i.e. $w_a(x) \leq \int_{-\infty}^{\infty} |f(x)|^p \mathrm{d}x \leq M\ \forall x \in \mathbb{R}$, and (3.41) with $\sigma = p\sigma$, we obtain

$$w_a(z) = \int_{-a}^{a} |f(x+\mathrm{i}y+t)|^p \mathrm{d}t \leq M\mathrm{e}^{p\sigma|y|}\,.$$

Subsequently we send a to infinity which finishes the proof. □

The following corollary is extensively used in the context of interpolation in the next sections.

Corollary 3.39. *For functions in Bernstein spaces $f \in \mathcal{B}^p_\sigma$ we have*

$$|f(x+\mathrm{i}y)| \leq C(p,\sigma)\|f\|_{\mathcal{B}^p_\sigma}\mathrm{e}^{\sigma|y|} \tag{3.46}$$

for $p \in [1,\infty]$ and $x, y \in \mathbb{R}$.

Proof. We start with the case $p \in [1,\infty)$. It is given by the Plancherel Pólya Theorem 3.38 that

$$\int_{-\infty}^{\infty} |f(x+\mathrm{i}y)|^p \mathrm{d}x \leq \mathrm{e}^{p\sigma|y|}\|f\|^p_{\mathcal{B}^p_\sigma}\,. \tag{3.47}$$

Because $|f|^p$ is a subharmonic function we further know by the mean-value property (3.35) that

$$|f(z)|^p \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z+r\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}\theta$$

for small enough r. Now integrate over the area rather than the circumference by integrating over $r\mathrm{d}r$ on both sides and we obtain for $z = x+\mathrm{i}y$

$$\begin{aligned}\pi|f(z)|^p &\leq \int_0^1\int_0^{2\pi} |f(z+r\mathrm{e}^{\mathrm{i}\theta})|^p r\mathrm{d}\theta\mathrm{d}r \leq \int_{-1}^1\int_{-\infty}^{\infty}|f(x+\mathrm{i}(y+s))|^p\mathrm{d}x\mathrm{d}s\\ &\leq 2\mathrm{e}^{\sigma p|y+s|}\|f\|^p_{\mathcal{B}^p_\sigma} \leq 2\mathrm{e}^{\sigma p(|y|+1)}\|f\|^p_{\mathcal{B}^p_\sigma}\,.\end{aligned}$$

Thus, we directly have for $p \in [1,\infty]$

$$|f(x+\mathrm{i}y)|^p \leq \frac{2}{\pi}\mathrm{e}^{\sigma p(|y|+1)}\|f\|^p_{\mathcal{B}^p_\sigma}\,.$$

Recall that for $p = \infty$ we have by (3.33), that $|f(x+\mathrm{i}y)| \leq M\mathrm{e}^{\sigma|y|}$ which completes the proof. □

3.3 Interpolation of entire functions

With Corollary 3.39 it is now possible to find an interpolation formula for $\mathcal{B}^p_\sigma$ with $p \in [1,\infty]$. For this purpose we have to introduce sine-type functions and discuss conditions on convergence of the infinite products $\prod_{n\in\mathbb{Z}}(1-\frac{z}{z_n})$ on compact sets. Using several fundamental factorization theorems and the fact that sine-type functions are Cartwright functions of class $\mathcal{C}$, we will see that such a representation can always be found to converge for sine type functions. We can use them to construct kernels similar to the Lagrangian interpolator for polynomials to find a unique interpolating function. Finally, we are able to show $\mathcal{B}^p_\sigma$ convergence for $p \in (1,\infty)$, whereas in the other cases we only have convergence in the topology of $\mathcal{S}'$.

3.3.1 Weierstrass and Hadamard Factorization theorem

First, we will look at some factorization theorems providin product representations for entire functions. Let us consider the infinite product $\prod_{n\in\mathbb{Z}}(1-\frac{z}{z_n})$ with $\lim z_n = \infty$. The product only converges absolutely if $\sum_{n\in\mathbb{Z}} \frac{1}{|z_n|} < \infty$. However, we can avoid such strict limitations by multiplying it by an exponential term.

Lemma 3.40. *The infinite product $\prod_{n\in\mathbb{Z}}(1-\frac{z}{z_n})\mathrm{e}^{p_n(z)}$ with the polynomial*

$$p_n(z) = \sum_{k=1}^{n} \frac{1}{k}\left(\frac{z}{z_n}\right)^k$$

converges uniformly on every bounded set of $\mathbb{C}$.

Proof. First, define the primary factors $E(u,p) = (1-u)\exp(\sum_{k=1}^{p}\frac{u^k}{k})$ for $p \in \mathbb{N}$ and complex numbers u. Because for $|u| < 1$ we have $\log(1+u) = \sum_{k=1}^{\infty}\frac{u^k}{k}(-1)^{k+1}$, we obtain for any $|u| < 1$

$$\log E(u,p) = -\sum_{k=p+1}^{\infty} \frac{u^k}{k}\,. \tag{3.48}$$

We then have for $|u| \leq \epsilon < 1$

$$|\log E(u,p)| \leq \sum_{k=p+1}^{\infty} |u|^k \leq \frac{|u|^{p+1}}{1-\epsilon}\,. \tag{3.49}$$

The second inequality follows from the simple calculation

$$\begin{aligned}\sum_{k=1}^{\infty}|u|^k - \sum_{k=1}^{p}|u|^k &= \frac{1}{1-|u|} - \frac{1-|u|^{p+1}}{1-|u|} \\ &\leq \frac{|u|^{p+1}}{1-\epsilon}\,.\end{aligned}$$

By choosing $u = \frac{z}{z_n}$ we have that $\sum_{n=1}^{\infty}\log E(\frac{z}{z_n}, n)$ converges uniformly on compact sets. This is because for every compact region you can find N such that $|z| < |z_n|$ fo all $n > N$ so that the series can be split into a finite and an infinite series. The infinite series is always convergent since if we denote $f_n(z) = |\log E(\frac{z}{z_n}, n)|$, we have by the choice of ϵ such that $|z/z_n| \leq \epsilon$ and (3.49)that

$$\sum_{n=N+1}^{\infty} \|f_n(z)\|_\infty \leq \left(\frac{1}{1-\epsilon}\right)^2 |\epsilon|^{N+2}$$

for all $N \geq 0$. Hence

$$\sum_{n=1}^{\infty}\log E(\frac{z}{z_n}, n) = \log\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right)\mathrm{e}^{p_n(z)}$$

is absolutely convergent and so is the infinite product. □

Theorem 3.41 (Weierstrass factorization)**.** *Every entire function f which has zeros $z_n \neq 0$ can be written as*

$$f(z) = z^m \mathrm{e}^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \mathrm{e}^{p_n(z)} \tag{3.50}$$

where $g(z)$ is some entire function and m is the multiplicity of the zero at $z = 0$.

Proof. Define $\phi(z) = z^m \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \mathrm{e}^{p_n(z)}$, which has the same zeros as f and is entire. Therefore, the quotient $f(z)/\phi(z) = \mathrm{e}^{g(z)}$ as it is entire and never zero. □

When the sequence of the inverse of the zeros form an ℓ^{p+1} sequence, the expression in (3.50) can be simplified.

Corollary 3.42. *Let f be an entire function with zeros $\{z_n\}$. If $\sum_{n=1}^{\infty} \frac{1}{|z_n|^{p+1}} < \infty$, then f can be represented as*

$$f(z) = z^m \mathrm{e}^{\tilde{g}(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\sum_{k=1}^{p} \frac{1}{k} \left(\frac{z}{z_n}\right)^k\right) \tag{3.51}$$

for some entire function $\tilde{g}(z)$.

Proof. First of all it is clear that $|f(z)|$ converges uniformly on compact sets, since

$$\sum_{n=1}^{\infty} \left|\log E\left(\frac{z}{z_n}, p\right)\right| \leq \frac{|z|^{p+1}}{1-\epsilon} \sum_{n=1}^{\infty} \frac{1}{|z_n|^{p+1}}$$

for $|z/z_n| \leq \epsilon < 1$. One can rewrite (3.51) as

$$f(z) = z^m \mathrm{e}^{\tilde{g}(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(p_n(z) - \sum_{k=p+1}^{n} \frac{1}{k} \left(\frac{z}{z_n}\right)^k\right) \tag{3.52}$$

By comparing (3.50) and (3.51) we have

$$\tilde{g}(z) = g(z) + \sum_{n=1}^{\infty} \sum_{k=p+1}^{n} \frac{1}{k} \left(\frac{z}{z_n}\right)^k \tag{3.53}$$

Using (3.49) it follows that for the infinite series where $|z/z_n| \leq \epsilon < 1$ that

$$\begin{aligned} \sum_{k=p+1}^{n} \frac{1}{k} \left|\frac{z}{z_n}\right|^k &\leq \sum_{k=p+1}^{\infty} \frac{1}{k} \left|\frac{z}{z_n}\right|^k \\ &\leq \frac{\left|\frac{z}{z_n}\right|^{p+1}}{1-\epsilon} \end{aligned}$$

so that by assumption $\sum_n \frac{1}{|z_n|^{p+1}} < \infty$, $\tilde{g}(z)$ as in (3.53) is actually an entire function which proves the corollary. □

Corollary 3.43. *If f is an entire function of exponential type, then we can write*

$$f(z) = z^m \mathrm{e}^{\tilde{g}(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \mathrm{e}^{z/z_n} .$$

Proof. Theorem 6 of 2.1.3. in [43] yields that for entire functions of finite order ρ with zeros $\{z_n\}$ the series $\sum_{n=1}^{\infty} \frac{1}{|z_n|^\alpha}$ converges whenever $\alpha > \rho$. Its proof uses Theorem 3 of 2.1.3. in [43] which states that $n(r) = O(r^{\rho+\epsilon})$, where $n(r)$ is the number of zeros with radius $|z_n| \leq r$. As functions of exponential type are of finite order $\rho = 1$, the corollary then follows with $\alpha = 2$ and $\rho = p = 1$ in Corollary 3.42. □

Theorem 3.44 (Hadamard factorization)**.** *If $f(z)$ is an entire function of finite order ρ with canonical factorization*

$$f(z) = z^m \mathrm{e}^{\tilde{g}(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\sum_{k=1}^{p} \frac{1}{k} \left(\frac{z}{z_n}\right)^k\right) , \tag{3.54}$$

then $\tilde{g}(z)$ is of polynomial degree at most ρ.

Proof. See [43] 2.5. Theorem 9. □

From Theorem 3.44 and Corollary 3.43 the following representation for entire functions of exponential type can be established.

Corollary 3.45. *When the entire function f is of finite order 1 and $f(0) \neq 0$, we have*

$$f(z) = f(0) \mathrm{e}^{cz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \mathrm{e}^{z/z_n} \tag{3.55}$$

uniformly convergent on compact subsets.

3.3.2 Cartwright functions

The goal is eventually to obtain a factorized representation of a sine-type function. Since a function of sine-type is of exponential type, we can readily represent it by the infinite product in (3.55). For sine-type functions however we can show that the exponential terms vanish. In order to see this, it is necessary to introduce the so-called Cartwright functions (Levin calls them functions of class $\mathcal{C}$) and the Cartwright-Levinson lemma.

Definition 3.46 (Cartwright functions)**.** A function is called a Cartwright function or function of class $\mathcal{C}$, whenever
(i) $f(z)$ is entire and of finite order 1 and
(ii) $\int_{-\infty}^{\infty} \frac{\log^+ |f(t)|}{1+t^2} \mathrm{d}t$ where $a^+ = \max(a, 0)$.

The following lemmata, stated without proof, are necessary to establish the Cartwright Levinson theorem.

Lemma 3.47. *Let $f(z)$ be entire in $\mathbb{C}_+$ and $\log|f(z)| \leq u(z)$ with $u(z)$ being a positive harmonic function and $\{a_k\}$ the zeros of $f(z)$. Then we have*

$$\sum_{k=1}^{\infty} \frac{\operatorname{Im} a_k}{1+|a_k|^2} < \infty \tag{3.56}$$

Proof. See [22] Lecture 14 Theorem 2. □

Lemma 3.48. *For every entire function of finite order 1 we have that*

$$\lim_{r\to\infty} \frac{n(r)}{r} < \infty \tag{3.57}$$

where $n(r)$ is the number of zeros with $|z_n| \leq r$.

Proof. See [43] Chapter 2.2 Theorem 3. □

Denote by $n_+(r,\alpha)$ and $n_-(r,\alpha)$ the number of zeros of the function $f(z)$ in the sectors $\{z : |z| \leq r, |\arg z| \leq \alpha\}$ and $\{z : |z| \leq r, |\arg z| \leq \pi - \alpha\}$ respectively. Now we can state the Cartwright-Levinson Theorem:

Theorem 3.49 (Cartwright, Levinson)**.** *Let $f \in \mathcal{C}$ and $\{a_k \neq 0\}$ its zero set. Then*
(i) $\sum_k \left|\operatorname{Im} \frac{1}{a_k}\right| < \infty$
(ii) $\lim_{r\to\infty} \frac{n_+(r,\alpha)}{r} = \lim_{r\to\infty} \frac{n_-(r,\alpha)}{r} = \frac{d}{2\pi}$
where d is the width of the indicator diagram as defined in [22]*.*
(iii) The limit $\lim_{R\to\infty} \sum_{|a_k|<R} \frac{1}{a_k}$ *is well-defined.*

Proof. To (i) see remark to [22] Lecture 14 Theorem 2.
To (iii): By the Schwartz-Nevanlinna representation (see [22] Section 2.2.) the sum can be rewritten as

$$\sum_{|a_k|<R} \frac{1}{a_k} = -\frac{f'(0)}{f(0)} + \frac{d}{2\pi}\int_0^{2\pi} |\sin\psi| \mathrm{e}^{-\mathrm{i}\psi} \mathrm{d}\psi + \sum_{|a_k|<R} \frac{\overline{a_k}}{R^2} + o(1) \tag{3.58}$$

Note that the second term in the sum vanishes and lets define $I_R\{\overline{a_k}\} := \sum_{|a_k|<R} \frac{\overline{a_k}}{R^2}$.

Now denote the zeros with $\operatorname{Re}(z) \geq 0$ by $a'_k = |a'_k|\mathrm{e}^{\mathrm{i}\psi'_k}$ and the others by a''_k. We first look at a'_k and obtain that

$$|I_R\{|a'_k|\} - I_R\{\overline{a'_k}\}| \leq \sum_{|a'_k|<R} \frac{|a'_k|^2}{R^2} \left|\frac{1-\mathrm{e}^{-\mathrm{i}\psi'_k}}{\sin\psi'_k}\right| \frac{|\sin\psi'_k|}{|a'_k|} \tag{3.59}$$

(i) implies that $\sum_k \frac{\sin\psi'_k}{|a'_k|}$ converges so that the left hand side of (3.59) vanishes as $R \to \infty$.

Additionally, setting $n_+(t) := n_+(t, \pi/2)$, we have through integration by parts

$$I_R\{|a_k'|\} = \frac{1}{R^2}\int_0^R t\mathrm{d}n_+(t) = \frac{n_+(R)}{R} - \frac{1}{R^2}\int_0^R n_+(t)\mathrm{d}t$$

And thus, according to (ii) and $n_+(t) = \frac{td}{2\pi} + o(t)$, we have

$$I_R\{|a_k'|\} = \frac{d}{2\pi} - \frac{d}{4\pi} + o(1) = \frac{d}{4\pi} + o(1)$$

Together with $I_R\{\overline{a_k''}\} = -\frac{d}{4\pi} + o(1)$ and $I_R\{\overline{a_k'}\} = I_R\{\overline{a_k'}\} + I_R\{\overline{a_k''}\} \to 0$ as $R \to \infty$ by (3.59), we obtain

$$\sum_{|a_k|<R} \frac{1}{a_k} = -\frac{f'(0)}{f(0)} + o(1)$$

which completes the proof. □

Corollary 3.50. *Whenever the assumptions of Theorem 3.49 hold and the a_k are ordered increasingly by their magnitude, we also have*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{a_k} = -\frac{f'(0)}{f(0)}$$

Proof. By Lemma 3.48 we have that

$$\Delta = \lim_{R\to\infty}\frac{n(R)}{R} < \infty\,. \tag{3.60}$$

By (ii) of Theorem 3.49 and defining $\Delta(R) = n(R)/R$, we can rewrite (3.60) to $R\Delta(R) + o(R) = n(R)$. We then see that

$$n(R + 0_+) - n(R) := \lim_{k\to 0} n(R+k) - n(R) = \lim_{k\to 0} k\Delta(R) + o(R) = o(R). \tag{3.61}$$

Now define $|a_i| =: R_i$ which is a monotonously increasing sequence and look at the series

$$\sum_{k=1}^{i}\frac{1}{a_k} = \sum_{k=1}^{n(R_i)}\frac{1}{a_k} - \sum_{k=n(R_i)+1}^{n(R_i)+n_0}\frac{1}{a_k} \tag{3.62}$$

with $n_0 < n(R_i + 0_+) - n(R_i)$. We know about the second term that

$$\sum_{k=n(R_i)+1}^{n(R_i)+n_0}\left|\frac{1}{a_k}\right| \leq \sum_{k=n(R_i)}^{n(R_i+0_+)}\frac{1}{R_i} \leq \frac{n(R_i+0_+) - n(R_i)}{R_i} \to 0$$

as $i \to \infty$ by (3.61). Thus, the second term of the sum in (3.62) converges to zero. Since $R_i \to \infty$ implies $n(R_i) \to \infty$, we have

$$\begin{aligned}\lim_{i\to\infty}\sum_{k=1}^{i}\frac{1}{a_k} &= \lim_{R_i\to\infty}\sum_{k=1}^{n(R_i)}\frac{1}{a_k}\\ &= \lim_{R\to\infty}\sum_{|a_k|<R}\frac{1}{a_k}\,.\end{aligned}$$

With (iii) of Theorem 3.49 the corollary follows. □

Corollary 3.51 (Product representation of sine-type functions)**.** *Every sine-type function can be written as*

$$f(z) = f(0)\lim_{n\to\infty}\prod_{|k|<n}\left(1-\frac{z}{z_k}\right) \tag{3.63}$$

for zeros $z_k \neq 0$ ordered by their magnitude. The infinite product converges uniformly on compact sets.

Proof. Starting from the (3.55) one sees immediately that $c = f'(0)/f(0)$. Now let

$$\begin{aligned}f(z) &= f(0)\lim_{R\to\infty}\exp\left(\left(c+\sum_{|z_k|<R}\frac{1}{z_k}\right)z\right)\prod_{|z_k|<R}\left(1-\frac{z}{z_k}\right)\\ &= f(0)\lim_{n\to\infty}\prod_{|k|<n}\left(1-\frac{z}{z_k}\right)\end{aligned}$$

As sine-type functions are also functions of class $\mathcal{C}$, we can use Corollary 3.50 so that the last equation follows from the fact that $c+\sum_{|k|<n}\frac{1}{z_k} = o(1)$ and $\mathrm{e}^{o(1)z} \to 1$ as $R \to \infty$. Note however that we only have conditional convergence (of ordering z_k according to their magnitudes) and uniform convergence only on compact sets. □

3.3.3 Interpolation series for $\mathcal{B}_\sigma^p$, $p \in (1,\infty)$

The sine-type functions with product representation as in (3.63) play a crucial role in interpolation in Berstein spaces. When given samples c_n a function f_a is said to satisfy the interpolation conditions if $f_a(\lambda_n) = c_n$ for all n. If the values are actually samples from a function f with $c_n = f(\lambda_n)$ the aim is usually that $f_a(z) = f(z)$ on the entire complex plane. This is the case whenever the interpolation condition only has one unique solution. Whether or not this is the case depends on the specific choice of λ_n. For $\mathcal{B}_\sigma^2$ sequences $\{\lambda_n\}$ for which this is true are called complete interpolating sequences. We will see that by choosing λ_n as zeros of a sine-type function, we even have a unique solution for $p \neq 2$. Reconstruction is then achieved by the interpolation formula $f_a(z) = \sum_n c_n\psi_n(z)$ with kernels which resemble the Lagrange interpolator and are of the form $\psi_n(z) = \prod_{m\neq n}\frac{z-\lambda_m}{\lambda_n-\lambda_m}$.

Theorem 3.52 (Irregular interpolation for $\mathcal{B}_\sigma^p$). *Let $S(z)$ be a sine-type function of type σ and let $\{\lambda_k\}_{k=-\infty}^{\infty}$ be its zero set. Then, the mapping*

$$f \mapsto \{f(\lambda_k)\}_{k=-\infty}^{\infty} \tag{3.64}$$

is an isomorphism between l^p and $\mathcal{B}_\sigma^p$ for each $p \in (1,\infty)$ with the inverse

$$\{c_k\}_{k=-\infty}^{\infty} \mapsto f(z) = \sum_{k=-\infty}^{\infty} c_k \frac{S(z)}{S'(\lambda_k)(z-\lambda_k)}\,. \tag{3.65}$$

The series on the right-hand side converges in the $\mathcal{B}_\sigma^p$-norm.

Remark. Define

$$F_k(z) = \frac{S(z)}{S'(\lambda_k)(z-\lambda_k)}\,. \tag{3.66}$$

Note that for $p=1$ we have uniform convergence in the supremum norm in a strip parallel to the real axis. This is because $\hat{x} \in \mathcal{B}_\sigma^1$ implies $\hat{x} \in \mathcal{B}_\sigma^p$ for $p>1$ and from (3.46) it follows for an arbitrary $H<\infty$

$$\sup_{z:|\operatorname{Im} z|<H} |\hat{x}(z) - \sum_{k=-N}^{N} \hat{x}(\lambda_k)F_k(z)| \le C\|\hat{x} - \sum_{k=-N}^{N} \hat{x}(\lambda_n)F_k\|_{\mathcal{B}_\sigma^p} \mathrm{e}^{\sigma|\operatorname{Im} z|}$$

with $p>1$. For these p we know from the theorem that the right hand side vanishes in the limit when $N\to\infty$. However it does not converge in the $\mathcal{B}_\sigma^1$ norm. The case $p=\infty$ has to be treated separately in the next section 3.3.5.

Proof. We reproduce the proof by [22] Lecture 22 Theorem 1.
If $f(z)$ is well-defined, the series in (3.65) satisfies the interpolation condition since by the representation in (3.63) we see that $F_k(z)$ has the form $\prod_{n\neq k}\frac{z-\lambda_n}{\lambda_k-\lambda_n}$. Thus given samples of a function $\{f(\lambda_k)\}$, the inverse in (3.65) actually yields the original function.

Now it remains to show that $f(z)$ is well-defined and that the mapping in (3.64) is isomorphic. Hardy spaces $\mathcal{H}_+^p$ include all functions which are analytic in $\mathbb{C}_+$ and for which the Hardy space norm

$$\|f\|_{\mathcal{H}_+^p} = \sup_{y>0}\int_{-\infty}^{\infty} |f(x+\mathrm{i}y)|^p \mathrm{d}x < \infty\,. \tag{3.67}$$

They have the property that

$$\sum_n |f(\lambda_n)|^p \le C\|f\|_{\mathcal{B}_\sigma^p}^p \tag{3.68}$$

if λ_n are zeros of a sine-type function (see [22] Lecture 9 Property 6.). Since for Bernstein spaces $\mathcal{B}_\sigma^p \subset \mathcal{H}_+^p$, this inequality is also valid for $f\in\mathcal{B}_\sigma^p$ such that the mapping $N : \mathcal{B}_\sigma^p \to l^p$ in (3.64) is bounded and thus continuous. Now we have to prove that the series in (3.65) converges for an arbitrary sequence $\{c_k\}_{k\in\mathbb{Z}}$ and the mapping (3.65) is bounded, i.e.

$$\|f\|_{\mathcal{B}_\sigma^p} \le \tilde{C} \sum_{k\in\mathbb{Z}} |c_k|^p \tag{3.69}$$

from which we can then readily conclude that N is injective.

Let all zeros $\{\lambda_k\}$ belong to a horizontal strip $\{z : 0 < \eta < \operatorname{Im} z < H < \infty\}$. Otherwise, we could use the same reasoning for a shifted function like $S(z + \mathrm{i}2H)$ where H is immanent to the sine type function as in (3.22). Then we know from the definition of sine-type functions and (3.23) that $0 < c < |S(x)| < C < \infty$ for $x \in \mathbb{R}$. Now define

$$\phi(z) = \sum_{k\in\mathbb{Z}} \frac{c_k}{S'(\lambda_k)(z-\lambda_k)} \tag{3.70}$$

for which the partial sums

$$\phi_{m,n}(z) = \sum_{k=m}^{n} \frac{c_k}{S'(\lambda_k)(z-\lambda_k)}$$

belong to the Hardy space $\mathcal{H}_-^p$. The norm in (3.67) is equivalent to the $\mathcal{L}^p$ norm on the real axis. This is because

$$\sup_y \int_{\mathbb{R}} |f(x+\mathrm{i}y)|^p \mathrm{d}x = \lim_{y\to 0} \int_{\mathbb{R}} |f(x+\mathrm{i}y)|^p \mathrm{d}x$$

by [22] Lecture 19 Theorem 1 and [16] Section II.2.3. A basic functional analytic result is that the norm of an element in a normed space $x \in X$ can always be calculated by

$$\|x\| = \sup_{0\neq f\in X^*} \frac{|f(x)|}{\|f\|_*}$$

The proof uses the Hahn Banach theorem. $\mathcal{H}_+^q$ is the dual space of $\mathcal{H}_-^p$ so that this result allows us to rewrite the $\mathcal{B}_\sigma^p$ norm of $\phi_{m,n}$ as follows

$$\|\phi_{m,n}\|_{\mathcal{B}_\sigma^p} = \|\phi_{m,n}\|_{\mathcal{H}_-^p} = K \sup\left\{\left|\int_{\mathbb{R}} \phi_{m,n}(x)\psi(x)\mathrm{d}x\right| : \psi \in \mathcal{H}_+^q,\ \|\psi\|_{\mathcal{H}_+^q} \le 1\right\}$$

where q is the conjugate exponent satisfying $1/p + 1/q = 1$ and K is a constant. Applying the residue theorem 3.10 gives

$$\|\phi_{m,n}\|_{\mathcal{H}_-^p} \le K \sup\left\{\left|\sum_{k=m}^{n} \frac{c_k}{S'(\lambda_k)}\psi(\lambda_k)\right| : \psi \in \mathcal{H}_+^q,\ \|\psi\|_{\mathcal{H}_+^q} \le 1\right\}$$

Now Hölders inequality together with Lemma 3.22 yields

$$\begin{aligned}\|\phi_{m,n}\|_{\mathcal{H}^p_-} &\le \tilde{K} \sup \left\{ \left(\sum_{k=m}^{n} |c_k|^p \right)^{1/p} \left(\sum_{k=m}^{n} |\psi(\lambda_k)|^q \right)^{1/q} : \psi \in \mathcal{H}^q_+,\ \|\psi\|_{\mathcal{H}^q_+} \le 1 \right\} \\ &\le \tilde{K} \left(\sum_{k=m}^{n} |c_k|^p \right)^{1/p} \qquad (3.71)\end{aligned}$$

because $\psi \in \mathcal{H}^q_+$ and thus (3.69) holds. As we now let $m \to -\infty$ and $n \to \infty$ and as (3.71) is valid for all $m, n \in \mathbb{R}$, we see that for (3.70), $\|\phi\|_{\mathcal{B}^p_\sigma} \le \tilde{K} \left(\sum_{k=-\infty}^{\infty} |c_k|^p\right)^{1/p}$ and thus

$$\begin{aligned}\|f\|_{\mathcal{B}^p_\sigma} &= \int_{-\infty}^{\infty} |\phi(x) S(x)|^p \mathrm{d}x \\ &\le B^p \|\phi\|_{\mathcal{B}^p_\sigma}\end{aligned}$$

where the inequality follows from Lemma 3.21. It follows that the infinite series in (3.69) converges uniformly and absolutely which completes the proof. □

Remark. Note that for $\sigma = \pi$ and $S(z) = \sin(\pi z)$ we obtain the well-known Whittaker-Shannon series for bandlimited functions

$$f(z) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin(\pi(z-k))}{\pi(z-k)} \qquad (3.72)$$

3.3.4 Interpolation for $\mathcal{B}^2_{T/2}$

For $p \in (1, \infty)$ Theorem 3.52 has shown that zeros of sine type functions allow for a unique solution of the interpolation condition. For $p = 2$ the same even holds for a superset of sequences, though it is not clear, how much bigger it is compared to the set of zeros of sine type functions.

Definition 3.53 (Complete interpolating sequence for $\mathcal{B}^2_\sigma$)**.** An infinite sequence is called complete interpolating for $\mathcal{B}^2_\sigma$ whenever the interpolation condition $f(\lambda_k) = c_k$ has a unique solution in $\mathcal{B}^2_\sigma$ for all $\{c_k\} \in l^2$, i.e. the mapping $N : f \mapsto \{f(\lambda_k)\}$ is an isomorphism.

Definition 3.54 (Riesz basis)**.** A system of elements $\{x_n\}$ in a Hilbert space $\mathcal{H}$ is called a Riesz basis if

$$\overline{\operatorname{span}}\{x_n\}_{n \in \mathbb{Z}} = \mathcal{H} \qquad (3.73)$$

and there exist positive constants A, B such that

$$A \|\{c_n\}\|^2_{\ell^2} \le \|\sum_n c_n x_n\|^2_{\mathcal{H}} \le B \|\{c_n\}\|^2_{\ell^2} \qquad \forall \{c_n\} \in \ell^2 \qquad (3.74)$$

A Riesz basis in $\{x_n\} \subset \mathcal{H}$ can be constructed by a linear bounded and bijective operator $T : \mathcal{K} \to \mathcal{H}$ upon an orthonormal basis $\{e_n\}$ from a separable Hilbert space $\mathcal{K}$ by $x_n = T e_n$.

Theorem 3.55. *To every Riesz basis $\{x_n\}$ in a Hilbert space $\mathcal{H}$ there exists a unique dual Riesz basis $\{y_n\}$ such that every $x \in \mathcal{H}$ can be represented by*

$$\begin{aligned} x &= \sum_{n\in\mathbb{Z}} \langle x, x_n\rangle y_n \\ &= \sum_{n\in\mathbb{Z}} \langle x, y_n\rangle x_n \end{aligned}$$

and (3.74) *holds for $\{y_n\}$ as well with constants $\frac{1}{B}$ and $\frac{1}{A}$. The dual basis can be constructed by $y_n = T^{*-1}e_n$ where the adjoint T^* of an operator T satisfies*

$$\langle x, T^* y\rangle = \langle Tx, y\rangle \,. \tag{3.75}$$

$\{y_n\}$ is the dual Riesz basis of the Riesz basis $\{x_n\}$ if and only if it is biorthogonal to $\{x_n\}$.

Proof. See [9] Sections 3.3 and 3.6. for all the statements except the last. The results there also show that for every basis there exists one unique biorthogonal system. Since a dual Riesz basis is biorthogonal to the original Riesz basis by construction and (3.75), the last statement follows. □

Theorem 3.56. *$\{\lambda_k\}_{k\in\mathbb{Z}}$ is a complete interpolating sequence if and only if $\{\mathrm{e}^{\mathrm{i}\lambda_k t}\}_{k\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{L}^2([-\sigma,\sigma])$.*

Proof. See [43] Section 4.5. Theorem 9. □

Theorem 3.57. *If $\{\lambda_k\}_{k\in\mathbb{Z}}$ are zeros of an entire function, then $\{\mathrm{e}^{\mathrm{i}\lambda_k t}\}_{k\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{L}^2([-\sigma,\sigma])$.*

Proof. Define

$$F_k(z) = \frac{S(z)}{S'(\lambda_k)(z-\lambda_k)} \tag{3.76}$$

with $S(z) = \lim_{n\to\infty} \prod_{|k|<n} \left(1 - \frac{z}{\lambda_k}\right)$. By Theorem 3.52 it directly follows that for each $f \in \mathcal{B}^2_\sigma$ we have $f(z) = \sum_k f(\lambda_k)F_k(z)$. As the mapping in (3.64) is isomorphic and bounded, we can use the Open Mapping Theorem (see [32] Vol.I Section III.10) to deduce that (3.74) holds for $\{F_k\}$, i.e. $\{F_k\}$ is a Riesz Basis for $\mathcal{B}^2_\sigma$.

Now using the Paley-Wiener theorem we can define a family of functions $\{\zeta_k(t) \in \mathcal{L}^2([-\sigma,\sigma])\}_k$ which fulfills

$$\int_{-\sigma}^{\sigma} \zeta_k(t)\mathrm{e}^{\mathrm{i}tz}\mathrm{d}t = F_k(z)\,.$$

Because of the isomorphism of the Fourier Laplace Transform, we have that $\{\zeta_k\}$ is a Riesz basis if and only if $\{\zeta_k\}$ is a Riesz basis for $\mathcal{L}^2([-\sigma,\sigma])$. We quickly see that $\{\mathrm{e}^{\mathrm{i}\lambda_n t}\}$ is the biorthogonal system to $\{\zeta_k\}$ since $F_k(\lambda_n) = \delta_{k,n}$. From Theorem 3.55 we can then conclude that $\{\mathrm{e}^{\mathrm{i}\lambda_n t}\}$ is a Riesz basis for $\mathcal{L}^2([-\sigma,\sigma])$ if and only if $\{\zeta_k\}$ is a Riesz basis for $\mathcal{L}^2([-\sigma,\sigma])$. □

Corollary 3.58. *If $\{\lambda_k\}_k$ is a complete interpolating sequence, then we can write for each $f \in \mathcal{B}^2_\sigma$*

$$f(z) = \sum_{k\in\mathbb{Z}} f(\lambda_k) F_k(z) \tag{3.77}$$

with F_k as in (3.76)*.*

One example for such a complete interpolating sequence are the zeros of sine-type functions which directly follows from Theorem 3.52. Note that for $p \neq 2$ we specifically need this smaller space in order to have convergence.

3.3.5 Interpolation of $\mathcal{B}^\infty_\sigma$ without oversampling

For functions in C_0 it has been shown that the Shannon series as in Theorem 3.52 converges uniformly on compact sets.

Theorem 3.59 (Moenich, Boche)**.** *Let $S(z)$ be a function of sine type σ, whose zeros $\{\lambda_k\}_k$ are all real and ordered increasingly. Furthermore, let F_k be defined as in* (3.76)*. Then, for all $T > 0$ and all $f \in \mathcal{B}^\infty_\sigma \cap C_0$ we have*

$$\lim_{N\to\infty} \max_{t\in[-T,T]} \left| f(t) - \sum_{k=-N}^{N} f(\lambda_k) F_k(t) \right| = 0\,. \tag{3.78}$$

Proof. See [29] Theorem 2. □

Remark. In [29] all results only considered the case of real zeros λ_n. However, the corresponding proofs can also be applied for complex zeros, using that all zeros of a sine-type function lie in a strip parallel to the real axis.

For general functions we can also show convergence on compact sets with a slightly different series. It is based on the uniform sampling version of Levin in [22] Lecture 21 and is, different than the Shannon series, absolutely convergent.

Theorem 3.60. *Let S be a sine-type function of type σ and let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be its zero set. For any sequence $\mathbf{c} = \{c_n\}_{n\in\mathbb{Z}} \in \ell^\infty$ there exists an entire function f of exponential type σ which solves the interpolation problem*

$$f(\lambda_n) = c_n\,, \quad n \in \mathbb{Z}\,. \tag{3.79}$$

Every such entire function admits the representation

$$f(z) = {\sum_{n\in\mathbb{Z}}}' c_n \frac{S(z)}{S'(\lambda_n)} \left[\frac{1}{z-\lambda_n} + \frac{1}{\lambda_n}\right] + C_s\, S(z) \tag{3.80}$$

with an arbitrary constant $C_s \in \mathbb{C}$ and where the sum converges uniformly and absolutely on compact subsets of $\mathbb{C}$.

Remark. The prime at the summation sign in (3.80) means that the second term in the braces is set to zero if $\lambda_n = 0$, and we will omit this prime subsequently.

For every $C_s \in \mathbb{C}$, the function f in (3.80) has the property that

$$|f(\xi + \mathrm{i}\eta)|\, \mathrm{e}^{-\sigma|\eta|} = o(|\xi + \mathrm{i}\eta|)\,, \quad \text{as } |z| \to \infty\,. \tag{3.81}$$

Theorem 3.60 basically states that for $p = \infty$ the interpolation problem $f(\lambda_k) = c_k$ is uniquely solvable by an entire function f of exponential type up to an additive sine-type term. However, it should be noted that the function defined in (3.80) may not be bounded on $\mathbb{R}$ for some sequences $\boldsymbol{c} \in \ell^\infty$, i.e. the left hand side of (3.80) may not be in $\mathcal{B}_\sigma^\infty$. For regular sampling points $\lambda_k = k$ Levin provides a sufficient condition in [22] when this is the case which we will also discuss in chapter 5.

Proof. The proof is partly along the same lines as in [22, Lect. 21, Theorem 1]. Let $\lambda_k = \xi_k + \mathrm{i}\eta_k\, \forall k$. First, it is shown that the sum in (3.80) converges. To this end, consider the partial sum

$$\varphi_{m,n}(z) = \sum_{|k|=n+1}^{m} c_k \frac{1}{S'(\lambda_k)} \left[\frac{1}{z-\lambda_k} + \frac{1}{\lambda_k}\right]\,.$$

Since S is a sine-type function, one knows that $\inf_{n\in\mathbb{Z}} |S'(\lambda_n)| =: C_0 > 0$ by Lemma 3.22. Then by the triangular and Cauchy-Schwarz inequality

$$\begin{aligned} |\varphi_{m,n}(z)| &\le \frac{\|\boldsymbol{c}\|_\infty}{C_0} \sum_{|k|=n+1}^{m} \left|\frac{1}{z-\lambda_k} + \frac{1}{\lambda_k}\right| \le \frac{\|\boldsymbol{c}\|_\infty}{C_0}\, |z| \sum_{|k|=n+1}^{m} \frac{1}{|\lambda_k|\,|z-\lambda_k|} \\ &\le \frac{\|\boldsymbol{c}\|_\infty}{C_0}\, |z| \left(\sum_{|k|=n+1}^{m} \frac{1}{|\lambda_k|^2}\right)^{1/2} \left(\sum_{|k|=n+1}^{m} \frac{1}{|z-\lambda_k|^2}\right)^{1/2}\,. \end{aligned} \tag{3.82}$$

Since the zeros of a sine-type function are separated, there exists a $d > 0$ such that $|\lambda_m - \lambda_n| > 2\,d$ for all $m \ne n$. If $\delta \in (0, d)$ is fixed and z lies outside the set

$$\bigcup_{n\in\mathbb{Z}} \{z \in \mathbb{Z} : |z - \lambda_n| \le \delta\}$$

then there exists one zero, say λ_0, which is closest to z and for which $|z - \lambda_0| > \delta$. For all other zeros λ_k we then have

$$|z - \lambda_k| \ge \left|(\lambda_k - \lambda_0) - d\frac{\lambda_k - \lambda_0}{|\lambda_k - \lambda_0|}\right| = |\lambda_k - \lambda_0| \left|1 - \frac{d}{|\lambda_k - \lambda_0|}\right| \ge \frac{1}{2}\, |\lambda_k - \lambda_0|$$

using for the last inequality that $|\lambda_k - \lambda_0| \ge 2d$. Therewith, we get

$$\begin{aligned} \sum_{|k|=n+1}^{m} \frac{1}{|z-\lambda_k|^2} \le \sum_{k\in\mathbb{Z}} \frac{1}{|z-\lambda_k|^2} &\le \frac{1}{\delta^2} + \frac{1}{4} \sum_{k\in\mathbb{Z}, k\ne 0} \frac{1}{|\lambda_k - \lambda_0|^2} \\ &\le \frac{1}{\delta^2} + \frac{1}{4} \sum_{k\in\mathbb{Z}, k\ne 0} \frac{1}{|\xi_k - \xi_0|^2}\,. \end{aligned}$$

Since $\{\lambda_n\}_{n\in\mathbb{Z}}$ are zeros of a sine-type function, we can define $H := \sup_{k\in\mathbb{Z}} |\eta_k|$. Now let $m := \frac{2H}{2d}$ denote the maximal number of zeros in the vertical strip $\{z : |\operatorname{Re} z - \operatorname{Re} \lambda_0| < d\}$. Because the sequence $\{\xi_n\}_{n\in\mathbb{Z}}$ is assumed to be ordered increasingly, one can easily see that

$$|\xi_{m(k-1)+j} - \xi_0| \geq \sqrt{3}dk$$

for $j = 1 \dots m$ and $k \in \mathbb{N}_+$, noting the geometry of the densest circle packing. One can now rewrite the last sum and obtain

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{|\xi_k - \xi_0|^2} &= \sum_{k=1}^{\infty} \sum_{j=1}^{m} \frac{1}{|\xi_{m(k-1)+j} - \xi_0|^2} \\ &\leq \sum_{k=1}^{\infty} \frac{m}{k^2 3d^2} = c \sum_{k=1}^{\infty} \frac{1}{k^2}\end{aligned}$$

with $c = \frac{m}{3d^2}$. The same holds for the sum over $k \in \mathbb{N}_-$. Hence we have that

$$\sum_{|k|=n+1}^{m} \frac{1}{|z - \lambda_k|^2} \leq \frac{1}{\delta^2} + \frac{c}{2} \sum_{k=1}^{\infty} \frac{1}{k^2} \quad =: M_\delta < \infty \tag{3.83}$$

with a constant M_δ which does not depend on z. Moreover, since the sequence $\{\lambda_n\}_{n\in\mathbb{Z}}$ is the zero set of a function of exponential type, the series $\sum_{n\in\mathbb{Z}} |\lambda_n|^{-2}$ is convergent (see proof of Corollary 3.43). Consequently, (3.82) and (3.83) imply that to every $\epsilon > 0$ there exists an $N(\epsilon) > 0$ such that

$$|\varphi_{m,n}(z)| \leq \epsilon\, |z| \quad \forall\, m, n > N(\epsilon)\ . \tag{3.84}$$

It follows that $\varphi_{m,0}(z)$ converges to a function $\varphi(z)$ uniformly on each compact subset of $\mathbb{C}$ as $m \to \infty$. We set $g(z) := S(z)\,\varphi(z)$. Since S is a sine-type function of type σ, it follows from (3.84) that g satisfies (3.81), and it is obvious that g solves the interpolation problem (3.79).

Now, let f be an entire function which satisfies (3.79) and (3.81). Then the same is also valid for the difference $f(z) - g(z)$. Since S is of exponential type σ, it follows that for $h(z) = [f(z) - g(z)]/S(z)$ we have $|h(z)| = o(|z|)$ as $|z| \to \infty$. By Liouville's theorem (see [35] Theorem 10.23.) $h(z)$ is a constant, which proves (3.80). □

We now offer another derivation and proof of convergence of the series (3.80). Especially, we can explicitly provide the constant C_s. By defining for $f \in \mathcal{B}_\sigma^\infty$ and any arbitrary $\zeta \notin \{\lambda_k\}$

$$g(z) = \frac{f(z) - f(\zeta)}{z - \zeta}$$

with $g(\zeta) = f'(\zeta)$, we have that $g \in \mathcal{B}_\sigma^2$ since f is holomorphic and thus has finite derivatives. Hence we can use Lemma 3.60 to obtain

$$g(z) = \sum_{n\in\mathbb{Z}} g(\lambda_n) \frac{S(z)}{S'(\lambda_n)(z - \lambda_n)}\ . \tag{3.85}$$

Again we will define the kernels $\psi_n(z) = \frac{S(z)}{S'(\lambda_n)(z-\lambda_n)}$. From (3.85) we obtain

$$f(z) = (z-\zeta)\sum_{n\in\mathbb{Z}} \frac{f(\lambda_n)}{\lambda_n - \zeta}\psi_n(z) + f(\zeta)\left[1 - (z-\zeta)\sum_{n\in\mathbb{Z}}\frac{1}{\lambda_n-\zeta}\psi_n(z)\right]. \quad (3.86)$$

We can see that the sequences $\{\frac{f(\lambda_n)}{\lambda_n-\zeta}\}$ and $\{\frac{1}{\lambda_n-\zeta}\}$ are in ℓ^2 because we have zeros of sine type. Furthermore $\sum_{n\in\mathbb{Z}} c_n\psi_n(z)$ for any $\{c_n\} \in \ell^2$ is absolutely convergent using Hölders inequality and Lemma 3.22. Thus the two infinite series (3.86) converge absolutely. From (3.86) we can also deduce that whenever $f_1, f_2 \in \mathcal{B}_\sigma^\infty$ coincide on $\{\lambda_n\}$ and ζ, $f_1(z) = f_2(z)\,\forall z \in \mathbb{C}$. If we now consider the function

$$\tilde{f}(z) = \frac{f(\zeta)}{S(\zeta)}S(z) + (z-\zeta)\sum_{n\in\mathbb{Z}}\frac{f(\lambda_n)}{\lambda_n-\zeta}\psi_n(z) \quad (3.87)$$

it is obvious that $\tilde{f} = f$ exactly at those points λ_n and ζ. Consequently, every $f \in \mathcal{B}_\sigma^\infty$ can be represented by (3.87). In particular for (3.80) with $\zeta = 0$ and

$$\frac{z}{\lambda_n(z-\lambda_n)} = \frac{1}{\lambda_n} + \frac{1}{z-\lambda_n}$$

we then have $C_s = \frac{f(\zeta)}{S(\zeta)}$.

3.3.6 Interpolation of $\mathcal{B}_\sigma^\infty$ with oversampling

When we apply oversampling we can show uniform convergence of non-vanishing functions on compact sets even with the Shannon series.

Theorem 3.61 (Moenich, Boche)**.** *Let $S(z)$ be a function of sine type σ, whose zeros $\{\lambda_k\}_k$ are all real and ordered increasingly. Furthermore, let F_k be defined as in* (3.76)*. Then, for all $T > 0$ and all $f \in \mathcal{B}_{\beta\sigma}^\infty$ with $0 < \beta < 1$ we have*

$$\lim_{N\to\infty}\max_{t\in[-T,T]}\left|f(t) - \sum_{k=-N}^{N} f(\lambda_k)F_k(t)\right| = 0 \quad (3.88)$$

Proof. See [29] Theorem 4. □

The following result applies Theorem 3.60 to the oversampling case which enables absolute convergence but also only on compact sets. The unknown additive term in (3.80) then vanishes and we obtain a reconstruction formula for functions in $f \in \mathcal{B}_\sigma^\infty$.

Theorem 3.62. *Let S be a sine-type function of type $\tilde{\sigma}$ and let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ be its zero set. If $\tilde{\sigma} > \sigma$ then*

$$v(z) = \sum_{n\in\mathbb{Z}} v(\lambda_n)\frac{S(z)}{S'(\lambda_n)}\left[\frac{1}{z-\lambda_n} + \frac{1}{\lambda_n}\right] \qquad \forall v \in \mathcal{B}_\sigma^\infty \quad (3.89)$$

where the sum converges absolutely and uniformly on compact subsets of $\mathbb{C}$. *Moreover, there exists a constant* C_u *such that*

$$\lim_{N\to\infty} \max_{t\in\mathbb{R}} \left| v(t) - \sum_{n=-N}^{N} v(\lambda_n) \frac{S(t)}{S'(\lambda_n)} \left[\frac{1}{t-\lambda_n} + \frac{1}{\lambda_n} \right] \right| \leq C_u \, \|v\|_{\mathcal{B}_\sigma^\infty} \tag{3.90}$$

for all $v \in \mathcal{B}_\sigma^\infty$.

Proof. Since $v \in \mathcal{B}_\sigma^\infty$ and Λ is the zero set of a sine-type function, the sequence $\{c_n = v(\lambda_n)\}_{n\in\mathbb{Z}}$ is in ℓ^∞. Because $v \in \mathcal{B}_\sigma^\infty \subset \mathcal{B}_{\tilde{\sigma}}^\infty$, we know from Lemma 3.60 that every entire function $\widetilde{v}$ of exponential type $\tilde{\sigma}$ which satisfies $\widetilde{v}(\lambda_n) = c_n$ for all $n \in \mathbb{Z}$ has the form $\widetilde{v}(z) = f(z) + C_s\, S(z)$ where $f(z)$ stands for the sum on the right hand side of (3.89). Consequently, also v has to have this form, i.e. $v(z) = f(z) + C_s\, S(z)$ and we have to prove that $C_s = 0$. To this end, it is sufficient to show that f is of exponential type σ. Then, also $v - f$ is of exponential type σ such that in the equation

$$v(z) - f(z) = C_s\, S(z), \tag{3.91}$$

the modulus of the left hand side can be upper bounded by

$$|v(z) - f(z)| \leq B_1 \mathrm{e}^{\sigma|z|} \tag{3.92}$$

for all sufficiently large $|z|$ and with a certain constant $B_1 > 0$. Now assume $C_s > 0$. Because S is of sine-type $\tilde{\sigma}$, the modulus of the right hand side in (3.91) can be lower bounded by $|C_s S(z)| \geq B_2 \exp(\tilde{\sigma}|z|)$ for all sufficiently large $|z|$ and with a constant $B_2 > 0$. But since $\tilde{\sigma} > \sigma$ this yields a contradiction to the upper bound in (3.92). It follows that $C_s = 0$, i.e. that (3.89) holds.

It remains to show that f is of exponential type σ. Without loss of generality, we assume $v(0) = 0$ (otherwise, one applies the following reasoning to $v(z) - v(0)$) and consider the function $g(z) := v(z)/z$. Since v is of exponential type σ, to every $\epsilon > 0$ there exists a constant $C(\epsilon)$ such that

$$|g(z)| = \frac{1}{|z|}\, |v(z)| \leq \frac{1}{|z|}\, C(\epsilon)\, \mathrm{e}^{(\sigma+\epsilon)|z|} \leq C(\epsilon)\, \mathrm{e}^{(\sigma+\epsilon)|z|} \quad \text{for all } |z| \geq 1 \,.$$

This shows that g is of exponential type σ. Moreover the restriction of g to $\mathbb{R}$ is square integrable so that $g \in \mathcal{B}_\sigma^2 \subset \mathcal{B}_{\tilde{\sigma}}^2$. Therefore Theorem 3.52 can be applied to g, which gives

$$g(z) = \frac{v(z)}{z} = \sum_{n\in\mathbb{Z}} \frac{v(\lambda_n)}{\lambda_n} \frac{S(z)}{S'(\lambda_n)} \frac{1}{z-\lambda_n} \,.$$

By multiplying the whole equation by z, the right hand side becomes equal to $f(z)$ and shows that f is of exponential type σ.

In order to prove (3.90), we assume without loss of generality that $0 \notin \Lambda$ and we write $(\mathrm{T}_N v)(t)$ for the finite sum in (3.90), and $(\mathrm{A}_N v)(t)$ for the finite Shannon series, i.e.

$$(\mathrm{A}_N v)(t) = \sum_{n=-N}^{N} v(\lambda_n) \frac{S(t)}{S'(\lambda_n)} \frac{1}{t - \lambda_n} \,, \quad t \in \mathbb{R} \,.$$

Both approximation series are obvious related by

$$(\mathrm{T}_N v)(t) = (\mathrm{A}_N v)(t) - \frac{(\mathrm{A}_N v)(0)}{S(0)} S(t) \,.$$

For $v \in \mathcal{B}_\sigma^\infty$, the triangle inequality yields

$$|v(t) - (\mathrm{T}_N v)(t)| \leq |v(t) - (\mathrm{A}_N v)(t)| + \left| \frac{(\mathrm{A}_N v)(0)}{S(0)} \right| |S(t)| \,.$$

Now it is known [29, Theorem 5] that the first term on the right hand side is uniformly bounded by $C_1 \|v\|_\infty$ for all $t \in \mathbb{R}$ and all $N \in \mathbb{Z}$ with a certain constant C_1 independent of t and N. The same result also implies that the second term is uniformly bounded by a constant of the form $C_2 \|v\|_\infty$ which is why (3.90) holds and the proof is complete. □

Remark. Note that in Theorem 3.60, the sequence $\{c_n\}_{n\in\mathbb{Z}} \in \ell^\infty$ was arbitrary. In this case however, it arises from sampling an entire function $v \in \mathcal{B}_{T'/2}^\infty$ so that the existence and boundedness of the solution in (3.80) is naturally given.

Lemma 3.63. *Given that $v_N, v \in \mathcal{L}^\infty \subset \mathcal{S}'$ $\forall N \in \mathbb{Z}$ and*
(i)

$$\lim_{N\to\infty} \max_{t\in\mathbb{R}} |v(t) - v_N(t)| \leq c_u \|v\|_{\mathcal{L}^\infty} \tag{3.93}$$

(ii)

$$\|v - v_N\|_{\mathcal{L}^\infty} < \epsilon \tag{3.94}$$

for all $N > N_\epsilon$,
we have that for $N \to \infty$

$$(v - v_N)(\phi) \to 0 \qquad \forall \phi \in \mathcal{S} \tag{3.95}$$

i.e. $v_N \to v$ as $N \to \infty$ in the topology of S'.

Proof. Let's assume that without loss of generality $\int_{t\in\mathbb{R}} |\phi(t)| \mathrm{d}t \leq 1$. Since $\phi \in \mathcal{S}$, ϕ vanishes at infinity and

$$\forall \epsilon \; \exists T > 0 : \int_{t\in[-\infty,-T]\cup[T,\infty]} |\phi(t)| \mathrm{d}t < \epsilon \tag{3.96}$$

Now let $\Omega := [-T, T]$. By (3.96) you can choose T such that $\int_{\mathbb{R}\setminus\Omega} |\phi(t)|\mathrm{d}t < \frac{\epsilon}{c_u \|v\|_{\mathcal{L}^\infty}}$ and choose N_ϵ such that (3.94) holds. Then we get for all $\phi \in \mathcal{S}$ that

$$\begin{aligned}(v - v_N)(\phi) &= \int_\Omega (v - v_N)\phi \mathrm{d}t + \int_{\mathbb{R}\setminus\Omega} (v - v_N)\phi \mathrm{d}t \\ &\leq \int_\Omega \epsilon\phi \mathrm{d}t + \int_{\mathbb{R}\setminus\Omega} c_u \|v\|_{\mathcal{L}^\infty} \phi \mathrm{d}t \leq 2\epsilon\end{aligned}$$

for all $N > N_\epsilon$ and the lemma follows. □

Remark. Unfortunately we cannot say that $\lim_{N\to\infty} v_N \in \mathcal{L}^\infty$ although $\lim_{N\to\infty}(v - v_N)(\phi) \to 0$. This is because the limit and the integral can in general not be exchanged since we do not have uniform convergence of $v_N \to v$ everywhere but only on compact sets.

The next corollary now immediately follows from Theorem 3.62 and Lemma 3.63.

Corollary 3.64. *The partial sum in Theorem 3.62*

$$v_N(z) = \sum_{n=-N}^{N} v(\lambda_n) \frac{S(z)}{S'(\lambda_n)} \left[\frac{1}{z - \lambda_n} + \frac{1}{\lambda_n}\right] \tag{3.97}$$

converges to v in the topology of $\mathcal{S}'$.

Interestingly, for vanishing functions, it is even possible to obtain uniform convergence in the entire domain:

Theorem 3.65 (Boche, Moenich)**.** *Let $S(z)$ be a function of sine type σ, whose zeros $\{\lambda_k\}_k$ are all real and ordered increasingly. Furthermore, let F_k be defined as* (3.76)*. Then, for all $T > 0$ and all $f \in \mathcal{B}^\infty_{\beta\sigma} \cap C_0$ with $0 < \beta < 1$ we have*

$$\lim_{N\to\infty} \max_{t\in\mathbb{R}} \left| f(t) - \sum_{k=-N}^{N} f(\lambda_k) F_k(t) \right| = 0 \tag{3.98}$$

Proof. See [29] Theorem 6. □

4

Phase retrieval in infinite dimensions

After establishing the mathematical background, we now return to the original phase retrieval problem. While scientists in optics have used heuristic methods to approach this problem, the signal processing community has recently joined the discussion by providing theoretical background for convergence and perfect reconstruction of algorithms. Until now, the work has been concentrating on phase retrieval in finite dimensions which we will first briefly summarize. We then show how to use these results to provide a method for the infinite dimensional case.

4.1 Finite dimensional phase retrieval

For finite dimensional phaseless reconstruction of a signal $x \in \mathbb{C}^K$, we model the samples as $c_m = |\langle a^{(m)}, x\rangle|^2$ for $m = 1 \ldots M$ with $S : \mathbb{C}^K \to \mathbb{C}^M$. The phase retrieval problem then reads

$$\begin{aligned} &\text{Find } x \in \mathbb{C}^K \\ &\text{s. t. } |\langle a^{(m)}, x\rangle|^2 = c^{(m)} \ \forall m = 1, \ldots, M \end{aligned} \tag{4.1}$$

In the following we will refer to $a^{(m)}$ as the measurement vectors. Since the trace is invariant under cyclic permutations, we can rewrite the feasibility condition by using

$$\begin{aligned} |\langle a^{(m)}, x\rangle|^2 &= \operatorname{trace}(a^{(m)*}x \cdot x^* a^{(m)}) \\ &= \operatorname{trace}(a^{(m)}a^{(m)*} \cdot xx^*) =: \mathcal{A}(Q_x) \end{aligned}$$

with $Q_x = xx^*$, so that (4.1) is equivalent to

$$\begin{aligned} &\text{Find } Q_x \in \mathbb{C}^{K\times K} \\ &\text{s.t. } \operatorname{trace}(a^{(m)}a^{(m)*}Q_x) = c^{(m)} \ \forall m = 1,\dots,M \\ &\qquad \operatorname{rank}(Q_x) = 1 \\ &\qquad Q_x \geq 0 \qquad\qquad . \end{aligned} \tag{4.2}$$

Here, the rank-1 matrix Q_x is recovered instead of x. The signal vector x is an eigenvector of Q_x and can be determined by eigenvalue decomposition. Note however that there are infinitely many eigenvectors of the form $x \cdot e^{i\theta_0}$ with $\theta_0 \in [0, 2\pi]$. Therefore exact recovery using Q_x is only possible up to a constant unimodular phase factor which poses some problems when shifting the framework to the infinite dimensional case which we will have to solve.

The question in phase retrieval is whether the mapping S from the signal to the samples is injective, i.e. how to choose $a^{(m)}$ and M. If we had the scalar products $\langle a^{(m)}, x\rangle$, then we could just choose $\{a^{(m)}\}$ as a basis and reconstruct x via the dual basis. In this case we only need as many measurements as we have dimensions, i.e. $M = K$. However it is much more complicated if we are only given the magnitude of the measurement.

4.1.1 Frame theoretic results

Balan in 2006 [2] was able to show that for the complex signal space $\mathbb{C}^K$, once you generically choose $M \geq 4K - 2$ vectors $\alpha^{(m)}$, the mapping S is injective. The corresponding necessary condition was $M \geq 2K$. Lately, attempts have been made to prove the conjecture that $4K - 4$ is sufficient for the complex case (see [5] and [3]). For real signal vectors the necessary and sufficient lower bound is $2K - 1$ by [2]. A more constructive rather than generic result was given 2009 in [1] which showed that the matrix $Q_x = xx^*$ can be explicitly determined from the samples by choosing $a^{(m)}$ such that the orthogonal projections P_m onto the subspace containing $a^{(m)}$ are projective 2-designs (see [1] Definition 3.1.). In order to get an intuition when this is the case, we will briefly recapture some fundamental definitions of frame theory and give two explicit examples of families of vectors $\{a^{(m)}\}$ for which phaseless reconstruction is possible.

Definition 4.1 (Frame)**.** A family of vectors $\{a^{(m)}\}_{m=1\dots M} \subset H$ is called a frame for a K-dimensional Hilbert space H if there exist $A, B > 0$ such that

$$A\|x\|_2^2 \leq \sum_{m=1\dots M} |\langle x, a^{(m)}\rangle|^2 \leq B\|x\|_2^2 \qquad \forall x \in H \tag{4.3}$$

Definition 4.2 (Uniform tight frames)**.** A frame $\{a^{(m)}\}_{m=1\dots M}$ for H is called an A-tight frame, if in (4.3) A = B. If in addition there exists $b > 0$ such that $\|a^{(m)}\| = b \ \ \forall m = 1\dots M$ then we call this family a uniform A-tight frame.

The constant b is explicitly given by the constants K, M, A (see [1]) using

$$b = \sqrt{\frac{AK}{M}}\,. \tag{4.4}$$

Definition 4.3 (2-uniform A-tight frame)**.** If a frame $\{a^{(m)}\}_{m=1\dots M}$ for H is uniform, A-tight and additionally fulfills

$$|\langle a_j, a_k\rangle| = c \qquad \forall j \neq k \tag{4.5}$$

then it is called a 2-uniform A-tight frame.

Again we can also explicitly calculate c (see again [1]) by

$$c = \frac{A}{M}\sqrt{\frac{K(M-K)}{M-1}}\,. \tag{4.6}$$

Theorem 4.4. *The number of vectors M in a tight 2-uniform frame for a K-dimensional complex Hilbert space H is bounded by $M \leq K^2$.*

Proof. See [1] Proposition 2.3. □

In the following whenever $M = K^2$, we call it a maximal 2-uniform M/K-tight frame.

Example 4.5. An example for such a maximal 2-uniform M/K-tight frame for $K = 2$ was explicitly given in [2] as

$$\boldsymbol{\alpha}^{(1)} = \begin{pmatrix} a \\ b \end{pmatrix},\ \boldsymbol{\alpha}^{(2)} = \begin{pmatrix} b \\ a \end{pmatrix},\ \boldsymbol{\alpha}^{(3)} = \begin{pmatrix} a \\ -b \end{pmatrix},\ \boldsymbol{\alpha}^{(4)} = \begin{pmatrix} -b \\ a \end{pmatrix}$$

with $a = \sqrt{\frac{1}{2}(1 - \frac{1}{\sqrt{3}})}$ and $b = \mathrm{e}^{\mathrm{i}5\pi/4}\sqrt{\frac{1}{2}(1 + \frac{1}{\sqrt{3}})}$.

With these preliminaries we can now state the main result for finite dimensional phase retrieval.

Theorem 4.6 (Reconstruction formula for 2-uniform M/K tight frames)**.** *Let H be a complex K-dimensional Hilbert space and $\{a^{(m)}\}_{m=1\dots M}$ a 2-uniform M/K-tight frame with $M = K^2$. Then the corresponding orthogonal projections $Q_{a^{(m)}} = a^{(m)}a^{(m)*}$form a projective 2-design. Given a vector $x \in H$, the associated rank-1 matrix $Q_x = xx^*$ is determined by*

$$Q_x = \frac{K(K+1)}{M} \sum_{m=1\dots M} |\langle x, a^{(m)}\rangle|^2 Q_{a^{(m)}} - \|x\|^2 I \tag{4.7}$$

Proof. See [1] Theorem 3.4. By Theorem 4.4 we know that the maximal number of vectors in a 2-uniform M/K-tight frame is given by $M = K^2$. □

Note that $\|a^{(m)}\| = 1$ for all $m = 1 \dots M$ because of (4.4) so that $Q_{a^{(m)}}$ correspond to the P_m which are the orthogonal projection on the one dimensional subspaces contaning $a^{(m)}$. Another example of a family of vectors which forms a projective 2-design is the union of mutually unbiased bases:

Definition 4.7 (Mutually unbiased bases)**.** A family of vectors $\{a_n^{(m)}\}_{m=1\dots M}^{n=1\dots K}$ of a K-dimensional Hilbert space H are said to form M mutually unbiased bases, if $\{a_n^{(m)}\}_{n=1\dots K}$ is a basis for each m and for different bases $m \neq m'$ we have

$$|\langle a_n^{(m)}, a_{n'}^{(m')}\rangle| = \frac{1}{\sqrt{K}} \qquad \forall\, n, n' = 1 \dots K \tag{4.8}$$

and within each basis $m = 1 \dots M$ it holds

$$|\langle a_n^{(m)}, a_{n'}^{(m)}\rangle| = \delta_{n,n'} \tag{4.9}$$

where $\delta_{n,n'}$ stands for the Kronecker-Delta which is equal to one for $n = n'$ and zero everywhere else. In other words, basis vectors belonging to the M different bases have inner products with fixed magnitude and vectors within a basis are orthonormal.

For any $K + 1$ mutually unbiased bases in $\mathbb{C}^K$ we can establish a similar analytical formula such as (4.7) (see [1]).

4.1.2 Optimization algorithms

Besides the above deterministic and frame-theoretic results, there have also been efforts to obtain conditions for perfect phaseless reconstruction by randomly choosing measurement vectors using optimization algorithms. First note that the phase retrieval problem in (4.2) is equivalent to the following minimization problem:

$$\begin{aligned} &\min_{Q_x}\ \operatorname{rank}(Q_x) \\ &\text{s. t. } \operatorname{trace}(\alpha^{(m)}\alpha^{(m)*}Q_x) = c^{(m)}\ \forall m = 1, \dots, M \\ &\qquad Q_x \geq 0 \end{aligned} \tag{4.10}$$

This is basically the first step in the PhaseLift process. Since problem (4.10) is NP-hard, in [7] the authors explored alternative semidefinite programs such as the trace-minimization

$$\begin{aligned} &\min_{Q_x}\ \operatorname{trace}(Q_x) \\ &\text{s. t. } \operatorname{trace}(\alpha^{(m)}\alpha^{(m)*}Q_x) = c^{(m)}\ \forall m = 1, \dots, M \\ &\qquad Q_x \geq 0 \end{aligned} \tag{4.11}$$

which empirically has a great performance though formal equivalence with (4.10) is only given under certain conditions which are generally invalid. The following log-det minimization has an objective function which is closer to the rank operator than the trace

$$\begin{aligned}&\min_{Q_x}\ \log(\det(Q_x+\epsilon I)) \\ &\text{s. t. } \operatorname{trace}(\alpha^{(m)}\alpha^{(m)*}Q_x)=c^{(m)}\ \forall m=1,\dots,M \\ &\quad Q_x\geq 0 \qquad\qquad\qquad\qquad .\end{aligned} \tag{4.12}$$

However the corresponding algorithms are computationally inefficient. A good heuristic for solving this problem is the iterative reweighting scheme

$$\begin{aligned}&\min_{Q_x}\ \operatorname{trace}(W_kQ_x) \\ &\text{s. t. } \operatorname{trace}(\alpha^{(m)}\alpha^{(m)*}Q_x)=c^{(m)}\ \forall m=1,\dots,M \\ &\quad Q_x\geq 0\end{aligned} \tag{4.13}$$

with

$$W_{k+1}=(X_k+\epsilon I)^{-1}\,.$$

However it is not a perfect heuristic, i.e. it is not guaranteed to find the global minimum because the objective functional is concave.

In [8] the authors have shown that when the number of measurements satisfies $M\geq c_0K\log K$, with c_0 sufficiently large, the trace minimization program (4.11) has a unique solution obeying $Q_x=xx^*$. This holds with probability $1-3\mathrm{e}^{-\gamma\frac{m}{n}}$ where γ is a positive absolute constant.

4.2 Sampling and Reconstruction in Bernstein spaces

Given the well-known results for the finite dimensional case, we were naturally interested in the question whether it is possible to extend the framework to the infinite dimensional case. Thakur [38] has been the only one to provide reconstruction methods for real-valued bandlimited functions in $\mathcal{B}^{\infty}_{T/2}$ by simply sampling the signal with a uniformly dense (similar to complete interpolating) sequence. He surprisingly even achieves a lower bound of only two times the Nyquist rate. However his proof was based on the real-valuedness of the signal and the results cannot be simply to the complex case which are our signals of interest in the following.

After investigating $\mathcal{B}^{2}_{T/2}$ in the paper [42], in this chapter I will outline the framework for the more general space of $\mathcal{B}^{p}_{T/2}$ with $p\in[1,\infty]$. The goal is mainly to find a measurement setup such that we can split our problem into two known subproblems: Finite dimensional phase retrieval as in section 4.1 and interpolation theory in section 3.3. Instead of a scalar product $\langle a^{(m)},x\rangle$ representing our measurement, in the inifinite dimensional case we will more generally have a linear measurement functional $x\mapsto\phi_n^{(m)}(x;\alpha,\lambda)$ which depends on certain parameters α and λ. The correct choice of α and λ in the setup enables the simplification of the problem and ensures that both subproblems have well-defined solutions, i.e. that the corresponding mappings are injective. Before we introduce the measurement setup, we need to first specify our signal space and establish the equivalence of reconstruction in the time and Fourier domain.

4.2.1 Signal space

Let $T > 0$ be a real number. $\mathbb{T} := [-T/2, T/2]$ stands for the closed interval of length T. Since our signals will be sampled in the frequency domain, we define our signal spaces in terms of their Fourier representations as follows: For any $1 \leq p \leq \infty$, our signal space will be the set

$$\mathcal{Y}^p_{T/2} := \{x \in \mathcal{S}' \ : \ \hat{x} = \mathcal{F}x \in \mathcal{B}^p_{T/2}\} \tag{4.14}$$

of all tempered distributions whose Fourier-Laplace transforms belong to the Bernstein space $\mathcal{B}^p_{T/2}$. In the following we call x the signal in the *time domain* and $\hat{x} = \mathcal{F}x$ the signal in the *Fourier domain.* According to the Paley-Wiener Theorem 2.23, every $x \in \mathcal{Y}^p_{T/2}$ has a compact support contained in the interval $\mathbb{T}$.

Note that for $p > 2$ our signal space $\mathcal{Y}^p_{T/2}$ contains tempered distributions which are *not* regular (cf. Sec. 2.2.3). while every $x \in \mathcal{Y}^p_{T/2}$ for $p \in (1, 2]$ is actually a function in $\mathcal{L}^{p'}(\mathbb{T})$.

4.2.2 Signal reconstruction in the Fourier and time domain

By Theorem 3.52 we know that each $\hat{x} \in \mathcal{B}^p_{T/2}$ has the following representation for $1 < p < \infty$ if Λ is the zero set of a sine-type function S

$$\hat{x}(z) = \sum_{n\in\mathbb{Z}} \hat{x}(\lambda_n) \frac{S(z)}{S'(\lambda_n)(z-\lambda_n)} \quad \text{for all } \hat{x} \in \mathcal{B}^p_{T/2} \tag{4.15}$$

where the sum converges in the norm of $\mathcal{B}^p_{T/2}$.

Note that the interpolation kernels

$$\hat{\psi}_n := \frac{S(z)}{S'(\lambda_n)(z-\lambda_n)} \tag{4.16}$$

themselves belong to $\mathcal{B}^p_{T/2}$ for every $1 < p < \infty$. Consequently, $\psi_n = \mathcal{F}^{-1}\hat{\psi}_n \in \mathcal{L}^2(\mathbb{T})$, and by the definition of $\hat{\psi}_n$ it can be seen that $\hat{\psi}_n \in C_0(\mathbb{R})$ for every $n \in \mathbb{Z}$. Recall that for $p = 1$ the series (4.15) converges in the supremum norm, uniformly on each horizontal strip in the complex plane, but not in the $\mathcal{B}^1_{T/2}$-norm. For $p = \infty$ the series in (4.15) converges uniformly on compact sets for $C_0 \cap \mathcal{B}^\infty_{T/2}$ without oversampling, and everywhere for the same function space with oversampling. In order to interpolate in the entire space $\mathcal{B}_\sigma^\infty$ we need the series (3.89) in Theorem 3.62 which converges in the topology of $\mathcal{S}'$.

By isometry of the Fourier Transform the Inverse Fourier Transform of both interpolation series also converge in the time domain.

Lemma 4.8. *Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ be the zero set of a sine-type function S of type $T/2$ and let $\psi_n = \mathcal{F}^{-1}\hat{\psi}_n$ be the inverse Fourier transforms of the functions defined in* (4.16)*. Then for $1 \le p < \infty$ we have*

$$x = \sum_{n\in\mathbb{Z}} \hat{x}(\lambda_n)\,\psi_n \qquad \forall\ x \in \mathcal{Y}_{T/2}^p \tag{4.17}$$

where the sum converges in the topology of $\mathcal{S}'$. Now let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ be the zero set of a sine-type function S of type $T'/2 > T/2$. Then (4.17) *also additionally holds for every $x \in \mathcal{Y}_{T/2}^\infty \cap \mathcal{L}^1(\mathbb{T})$. If we then choose*

$$\hat{\psi}_n = \frac{S(z)}{S'(\lambda_n)}\left(\frac{1}{z-\lambda_n} + \frac{1}{\lambda_n}\right), \tag{4.18}$$

the series (4.17) *with $\psi_n = \mathcal{F}^{-1}\hat{\psi}_n$ converges for every $p \in [1,\infty]$, especially also for $\mathcal{Y}_{T/2}^\infty \setminus \mathcal{L}^1(\mathbb{T})$.*

Proof. Let $\hat{x}_N := \sum_{n=-N}^N \hat{x}(\lambda_n)\,\hat{\psi}_n$, then for $p \in (1,\infty)$, $\|\hat{x} - \hat{x}_N\|_{\mathcal{B}_{T/2}^p} \to 0$ as $N \to \infty$ for all $\hat{x} \in \mathcal{B}_{T/2}^p$ by Theorem 3.52. Using (3.46) it follows that

$$\begin{aligned}\left|\int_{\mathbb{R}} [\hat{x}(\omega) - \hat{x}_N(\omega)]\,\phi(\omega)\,\mathrm{d}\omega\right| &\le \|\hat{x} - \hat{x}_N\|_\infty \left|\int_{\mathbb{R}} \phi(\omega)\,\mathrm{d}\omega\right| \\ &\le C\,\|\hat{x} - \hat{x}_N\|_{\mathcal{B}_{T/2}^p} \quad \text{for all } \phi \in \mathcal{S}\end{aligned}$$

with a constant C which only depends on p, T and ϕ. When $p = 1$, only the first inequality holds from which we can directly conclude the claim using the remark of Theorem 3.52. Consequently, $\hat{x}_N \to \hat{x}$ in $\mathcal{S}'$ for $p \in [1,\infty)$, and since $\mathcal{F}^{-1}$ is a continuous isomorphic mapping of $\mathcal{S}'$ onto $\mathcal{S}'$ (see, e.g., [40]) it follows that $x_N \to x$ in $\mathcal{S}'$ for all $x \in \mathcal{Y}_{T/2}^p$. For all regular distributions $x \in \mathcal{Y}_{T/2}^\infty \cap \mathcal{L}^1(\mathbb{T})$ we have $\hat{x} \in \mathbb{C}_0 \cap \mathcal{B}_{T/2}^\infty$. By applying oversampling, Theorem 3.65 yields uniform convergence $\|\hat{x} - \hat{x}_N\|_\infty \to 0$ and the result follows. For non-regular tempered distributions $x \in \mathcal{Y}_{T/2}^\infty \setminus \mathcal{L}^1(\mathbb{T})$ we explicitly have by Corollary 3.64 that $\hat{x}_N \to \hat{x}$ in $\mathcal{S}'$ and therefore also $x_N \to x$ in $\mathcal{S}'$. □

Remark. In the following we will often use Lemma 4.8 for reconstruction in our signal space $\mathcal{Y}_{T/2}^p$. Note however that it only gives us convergence in the topology of $\mathcal{S}'$ in general. The problem lies in the fact that the Fourier Transforms for $\mathcal{L}^p$ are not bounded for $p > 2$ as discussed in section 4.2.1. However for regular distributions such as $x \in \mathcal{L}^{p'}(\mathbb{T}) \cap \mathcal{Y}_{T/2}^p$ with $1 < p \le 2$ (i.e. $2 < p' < \infty$), the Hausdorff Young

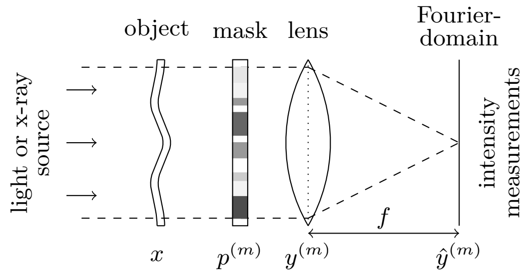


Figure 4.1: Schematic setup for structured modulation in optics using spatial light modulators (masks).

Theorem 2.20 together with Theorem 3.52 implies convergence in the $\mathcal{L}^{p'}$-norm. This does not hold for $p = 1$ although the Fourier Transform is bounded, because the partial sum in the Fourier domain (3.65) does not converge in the $\mathcal{B}^1_\sigma$ norm.

Example 4.9. Consider again the zeros of the special sine-type function $S(z) = \sin(\frac{T'}{2}z)$ of type $T'/2$ which are given by $\lambda_n = n\frac{2\pi}{T'}$, $n \in \mathbb{Z}$. The corresponding functions (4.16) are then $\hat{\psi}_n(z) = \operatorname{sinc}(\frac{T'}{2}[z - n\frac{2\pi}{T'}])$ where $\operatorname{sinc}(x) := \sin(x)/x$. (4.15) then becomes the Whittaker-Shannon sampling series, and it converges for all $\hat{x} \in \mathcal{B}^p_{T/2}$ with $1 < p < \infty$ and $T \leq T'$. The corresponding time-domain series (4.17) then becomes $x(t) = \sum_{n\in\mathbb{Z}} \hat{x}(\lambda_n)\, \mathrm{e}^{-\mathrm{i}n\frac{2\pi}{T'}t}$ for all $t \in \mathbb{T}$.

4.3 Measurement Methodology

The goal in this section is to find a measurement setup which makes $\hat{x} \mapsto |\phi^{(m)}_n(\hat{x})|$ injective. Inspired by the heuristic methods by scientists in optics, we apply a measurement methodology which uses oversampling in connection with structured modulations of the desired signal. Suppose $x \in \mathcal{Y}^p_{T/2}$ is the signal of interest. Although the loss of phase information might be intrinsic to the measurement procedure, it is often possible to influence the desired signal before the actual measurement. In optical applications one may apply spatial light modulators for these purposes. An example of a corresponding measurement setup (see also, e.g., [7, 21, 44]) is sketched in Fig. 4.1. There, the object of interest is illuminated by a coherent light source which produces a diffracted light field intrinsic to the object. In the case of X-ray crystallography it contains information about the electron density which is what we call the signal. Adding a spatial light modulator directly behind the object results in a modified signal $y^{(m)}(t) = x(t)\, p^{(m)}(t)$. The effective signal which reaches the detector is the Fourier Transform of $y^{(m)}$. The lense in Fig. 4.1 is used to project the far field to close range. There, the intensity is measured $|\hat{y}^{(m)}(\omega)|^2 = |(\mathcal{F}y^{(m)})(\omega)|^2$ and sampled at discrete points $\omega_n = n\beta$ with frequency sampling β. These measurements

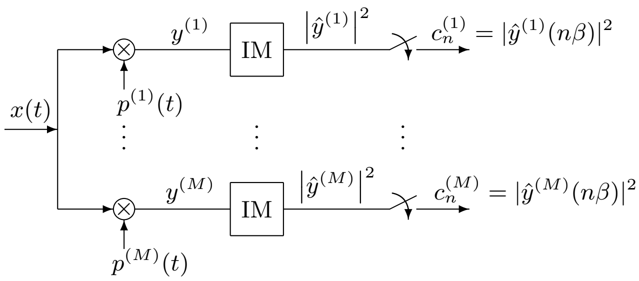


Figure 4.2: Measurement setup: In each branch, the unknown signal x is modulated with a different sequence $p^{(m)}$, $m = 1, 2, \ldots, M$. Subsequently, the intensities of the resulting signals $y^{(m)}$ are measured (IM) and uniformly sampled in the frequency domain.

are repeated with different masks $p^{(m)}$, $m = 1, \ldots, M$.

We will use this setup in the exact same way (see measurement scheme in Fig. 4.2) while specifically choosing $p^{(m)}$ to have the following general form

$$p^{(m)}(t) = \sum_{k=1}^{K} \overline{\alpha_k^{(m)}} e^{i\lambda_k t} . \tag{4.19}$$

The complex coefficients λ_k and $\alpha_k^{(m)}$ can be configured in a specific way in order to guarantee signal recovery. The samples in the mth branch are then given by

$$\begin{aligned} c_n^{(m)} &= |\hat{y}^{(m)}(n\beta)|^2 = \left| \sum_{k=1}^{K} \overline{\alpha_k^{(m)}} \, \hat{x}(n\beta + \lambda_k) \right|^2 \\ &= |\langle \hat{\mathbf{x}}_n, \boldsymbol{\alpha}^{(m)} \rangle|^2 , \qquad n \in \mathbb{Z} \end{aligned} \tag{4.20}$$

with the length K vectors

$$\boldsymbol{\alpha}^{(m)} := \begin{pmatrix} \alpha_1^{(m)} \\ \vdots \\ \alpha_K^{(m)} \end{pmatrix} \qquad \text{and} \qquad \hat{\mathbf{x}}_n := \begin{pmatrix} \hat{x}(n\beta + \lambda_1) \\ \vdots \\ \hat{x}(n\beta + \lambda_K) \end{pmatrix} .$$

Note that for this measurement setup our linear measurement functional reads $\phi_n^{(m)} = \langle \alpha^{(m)}, \hat{x}_n \rangle$ and for each n we have a finite dimensional phase retrieval problem. Now define $\lambda_{n,k} := n\beta + \lambda_k : n \in \mathbb{Z}\ , k = 1, \ldots, K$. Since recovery of the finite dimensional signals $\hat{x}_n$ is only possible up to a constant phase which is specific to each sampling instant n, the second step is to find $\{\hat{x}(\lambda_{n,k})\}_{n,k}$ given $\{\hat{x}_n e^{i\theta_n}\}_{n\in\mathbb{Z}}$. After successfully obtaining $\{\hat{x}(\lambda_{n,k})\}_{n,k}$, the remaining task is to recover x using the series in Lemma 4.8, which is equivalent to finding a unique interpolating function which passes through these sampling points.

Next we will first use the results in Section 4.1.1 to obtain conditions on $\alpha^{(m)}$ so that $\hat{x}_n \mapsto |\langle\alpha^{(m)}, \hat{x}_n\rangle|$ is injective. Then we address the particular choice of $\{\lambda_{n,k} := n\beta + \lambda_k : n \in \mathbb{Z}\,, k = 1, \ldots, K\}$ which is sufficient to guarantee uniqueness of the interpolation.

4.3.1 Choice of the coefficients $\boldsymbol{\alpha}_k^{(m)}$

In order to determine the vector $\hat{\mathbf{x}}_n \in \mathbb{C}^K$ from the M intensity measurements (4.20), we need to choose $\{\boldsymbol{\alpha}^{(1)}, \ldots, \boldsymbol{\alpha}^{(M)}\}$ as a 2-uniform K-tight frame and $M = K^2$ using the results from Section 4.1.1. This yields our first condition on $\alpha^{(m)}$

Condition A. A sampling system as in Fig. 4.2 is said to satisfy Condition A if the coefficients $\alpha_k^{(m)}$ in (4.19) are such that $\{\boldsymbol{\alpha}^{(1)}, \ldots, \boldsymbol{\alpha}^{(M)}\}$ constitutes a 2-uniform M/K-tight frame and $M = K^2$.

Using Theorem 4.6, reconstruction will then be based on the following formula

$$Q_{\hat{\mathbf{x}}_n} = \frac{(K+1)}{K} \sum_{m=1}^{M} c_n^{(m)} Q_{\boldsymbol{\alpha}^{(m)}} - \frac{1}{K} \sum_{m=1}^{M} c_n^{(m)} I \tag{4.21}$$

with rank-1 matrices $Q_{\mathbf{x}} = \mathbf{x}\mathbf{x}^*$. The last term results from the property that the frame is M/K-tight, such that

$$\|x\| = \frac{K}{M} \sum_{m=1 \ldots M} |\langle x, a^{(m)}\rangle|^2 \tag{4.22}$$

Note that in principle we could choose any other frame for which the resulting mapping $x \mapsto c_n^{(m)}$ is injective, but then we would not be able to give the needed number measurements M in terms of K nor have an analytical reconstruction formula like (4.21) which is later used for the discussion of robustness in Chapter 5.

4.3.2 Choice of the interpolation points

Let $\{\lambda_k\}_{k=1}^K$ be ordered increasingly by their real parts. For each $n \in \mathbb{Z}$, the vector $\hat{\mathbf{x}}_n$ contains the values of $\hat{x}$ at K distinct interpolation points in the complex plane. The set of these interpolation points for each n is denoted by

$$\boldsymbol{\lambda}_n^a := \{\lambda_{n,k}^a\}_{k=1}^K \quad \text{with} \quad \lambda_{n,k}^a = n\beta + \lambda_k\ , \quad n \in \mathbb{Z}\ . \tag{4.23}$$

Therein, the parameter $a \in \mathbb{N}$ denotes the number of overlapping points of consecutive sets (cf. also Fig. 4.3). We need a to be at least one, so that we can achieve for the constant phases that $\theta_n = \theta_0$ for all $n \in \mathbb{Z}$. The overlap condition can be formally written as

$$\lambda_{n,i}^a = \lambda_{n-1,K-i+1}^a \qquad \forall i = 1, \ldots, a\ . \tag{4.24}$$

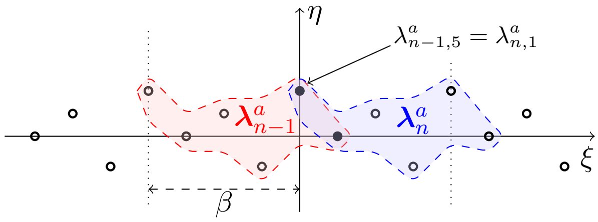


Figure 4.3: Illustration for the choice of interpolation points in the complex plane for $K = 6$ in (4.19) and an overlap $a = 2$.

In the following $\Lambda^a_{O,n} = \boldsymbol{\lambda}^a_n \cap \boldsymbol{\lambda}^a_{n+1}$ denotes the set of overlapping interpolation points between $\boldsymbol{\lambda}^a_n$ and $\boldsymbol{\lambda}^a_{n+1}$, and we define the overall interpolation sequences

$$\Lambda^a := \bigcup_{n\in\mathbb{Z}} \boldsymbol{\lambda}^a_n \,. \tag{4.25}$$

In general we allow for $a \geq 1$, but we will see that $a = 1$ is generally sufficient for reconstruction. Given a set $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ we will use the short notation $\hat{x}(\Lambda) = \{\hat{x}(\lambda_n)\}_{n\in\mathbb{Z}}$. As explained in Sec. 4.2.2, $x \in \mathcal{Y}^p_{T/2}$ can be perfectly reconstructed by (4.17) if Λ^a is the set of zeros of a sine-type function of type $T'/2 \geq T/2$ (for $p = \infty$ we need strict inequality). This results in the following condition for the interpolation points $\{\lambda_k\}$.

Condition B. A sampling system as in Fig. 4.2 is said to satisfy Condition B if for $p \in [1, \infty)$ the coefficients $\{\lambda_k\}_{k=1}^K$ in (4.19) are such that Λ^a as in (4.25) is the set of zeros of a sine-type function of type $T'/2 \geq T/2$ and satisfies (4.24) for a certain $1 \leq a < K$. For $p = \infty$ strict inequality has to hold whereas for $p = 2$ it is even sufficient for $\{\lambda_k\}_{k=1}^K$ to be a complete interpolating sequences as in Definition 3.53.

Interpolating sequences Λ^a fulfilling Condition B are β-periodic. Thus one specific way to obtain such sequences is to choose Λ^a as the set of zeros of a known β-periodic sine-type function of type $\tilde{T}/2 \geq T/2$. Based on such a zero set one may modify the imaginary part of the individual interpolation points or one can move the individual interpolation points slightly without changing the complete interpolating property and such that β-periodicity is preserved (see Section 3.1.5). So apparently, it is possible to construct many non-uniform complex interpolation sequences Λ^a which satisfy Condition B. One particularly simple construction is obtained by starting with the zeros of the sine-type function $\sin(\frac{\tilde{T}}{2}z)$, which has equally spaced zeros on the real axis (cf. Example 4.9) with β being an integer multiple of $4\pi/\tilde{T}$.

4.4 Phaseless Signal Recovery

We assume a sampling scheme as described in Section 4.3 (cf. Fig. 4.2) which satisfies Condition A and B. For this setup, we show that generically every $x \in \mathcal{Y}^p_{T/2}$ with $1 \leq p \leq \infty$ can be reconstructed from the samples in (4.20). The proof provides an explicit algorithm for signal recovery. In principle, it consists of three steps. First, a finite block of K samples of the Fourier domain signal $\hat{x}$ is reconstructed up to a constant phase factor from the $m = 1, \dots, M$ intensity measurements (4.20) taken at each sampling instant n. For this purpose $\alpha^{(m)}$ must fulfill Condition A, so that we can use finite dimensional phaseless reconstruction like formula (4.7). In the second step we exploit that by our construction of the interpolation points in Condition B, consecutive blocks have at least an overlap. Therewith, it is possible to make the unimodular factors θ_n in each block n constant over all blocks. In the last step the original function is recovered using Lemma 4.8.

Theorem 4.10. *Let $x \in \mathcal{Y}^p_{T/2}$ with $1 \leq p \leq \infty$ be sampled according to the scheme in Sec. 4.3 which satisfies Condition A and B, and let $\mathbf{c} = \{c_n^{(m)} : m = 1, \dots, M; n \in \mathbb{Z}\}$ be the sampling sequence in* (4.20)*. If the set $\hat{x}(\Lambda^a_{O,n})$ contains at least one non-zero element for each $n \in \mathbb{Z}$, then x can be perfectly reconstructed from $\mathbf{c}$ up to a constant phase.*

Proof. According to Condition B of the sampling system and Theorem 3.52, x can be reconstructed from the vectors $\{\hat{\mathbf{x}}_n\}_{n\in\mathbb{Z}}$ using (4.17) where the choice of the kernels depends on p. It remains to show that $\{\hat{\mathbf{x}}_n\}_{n\in\mathbb{Z}}$ can be determined from $\boldsymbol{c}$.

Let $n \in \mathbb{Z}$ be arbitrary. Since the sampling system satisfies Condition A, we can use (4.21) to obtain the rank-1 matrix $Q_n := \hat{\mathbf{x}}_n \hat{\mathbf{x}}_n^*$ from the measurements $\{c_n^{(m)}\}_{m=1}^M$. Then $\hat{\mathbf{x}}_n \in \mathbb{C}^K$ can be recovered by factorizing Q_n e.g. using eigenvalue decomposition. However, such a factorization is only unique up to a constant phase factor θ_n although the phase differences $\Delta\phi_{n,i}$ and magnitudes $|[\hat{\mathbf{x}}_n]_i|$ are precise. If we denote by $\phi^0_{n,i}$ the correct phase of $\hat{x}(\lambda^a_{n,i})$, then $\phi^0_{n,i} = \phi_{n,i} + \theta_n$ for all $i = 1 \dots K$ and $\phi_{n,i} = \phi_{n,1} + \Delta\phi_{n,i}$ as well as $\phi^0_{n,i} = \phi^0_{n,1} + \Delta\phi_{n,i}$ with $\Delta\phi_{n,i} = \arg([Q_n]_{i,1})$ for all n, i. If the constant phase θ_n for one sampling instant n is known, the vector $\hat{\mathbf{x}}_n \mathrm{e}^{\mathrm{i}\theta_n}$ can be analytically determined from Q_n by

$$\hat{x}(n\beta + \lambda_k)\mathrm{e}^{\theta_n} = \sqrt{[Q_n]_{k,k}}\,\mathrm{e}^{\mathrm{i}(\phi_{n,1}+\Delta\phi_{n,i})},\ \forall k \neq i\,. \tag{4.26}$$

The goal is to have $\phi^0_{n,i} = \phi_{n,i} + \theta_0$ for all $n \in \mathbb{Z}$ and $i = 1 \dots K$. Assume that we start the recovery of the sequence $\{\hat{\mathbf{x}}_n\}_{n\in\mathbb{Z}}$ at a certain $n_0 \in \mathbb{Z}$. In this initial step, we set the constant phase of $\hat{\mathbf{x}}_{n_0}$ arbitrarily to $\theta_0 \in [-\pi, \pi]$. In the next step, we determine $\hat{\mathbf{x}}_{n_0+1}$. After the factorization of Q_{n_0+1}, the vector $\hat{\mathbf{x}}_{n_0+1}$ is only determined up to a constant phase. However, since Λ^a_{O,n_0} is non-empty, and because $\hat{x}(\Lambda^a_{O,n_0})$ contains at least one non-zero element, we have phase knowledge of at least one entry of $\hat{\mathbf{x}}_{n_0+1}$, say $\hat{x}(\lambda^a_{n_0+1,i})$, where $\lambda^a_{n_0+1,i} = \lambda^a_{n_0,k}$ is an overlapping interpolation point of

$\boldsymbol{\lambda}_{n_0}^a$ and $\boldsymbol{\lambda}_{n_0+1}^a$. By requiring $\phi_{n_0+1,i} \stackrel{!}{=} \phi_{n_0,k}$, i.e. $\phi_{n_0+1,i}^0 \stackrel{!}{=} \phi_{n_0,k}^0 + (\theta_{n_0+1} - \theta_{n_0})$, we obtain $\theta_{n_0+1} = \theta_{n_0}$ so that the constant phase is passed on. Hence we will later refer to this step as "phase propagation". Using phase propagation, $\hat{\mathbf{x}}_{n_0+1}\mathrm{e}^{\mathrm{i}\theta_0}$ and successively all $\hat{\mathbf{x}}_n\mathrm{e}^{\mathrm{i}\theta_0}$ with $n = n_0 \pm 1, n_0 \pm 2, \ldots$ can then be determined by (4.26) which yields $\hat{x}(\Lambda^a)\,\mathrm{e}^{\mathrm{i}\theta_0}$. Note that the arbitrary setting of the phase of the initial vector $\hat{\mathbf{x}}_{n_0}$ results in a constant phase shift θ_0 for all $\hat{\mathbf{x}}_n$ which persists after the reconstruction of the time signal by (4.17).

□

Let us quickly recall what kind of convergences we have. The most useful convergence is $\mathcal{L}^p(\mathbb{T})$ convergence which is indeed given for regular distributions $x \in \mathcal{L}^{p'}(\mathbb{T}) \cap \mathcal{Y}_{T/2}^p$ for $1 < p \leq 2$ and $2 \leq p' < \infty$ correspondingly. For $\mathcal{Y}_{T/2}^p$ with $2 < p \leq \infty$ we only have convergence in the weak topology of $\mathcal{S}'$.

Theorem 4.10 states that $x \in \mathcal{Y}_{T/2}^p$ can only be reconstructed if $\hat{x} = \mathcal{F}x \in \mathcal{B}_{T/2}^p$ has at most $a-1$ zeros on the overlapping interpolation sets $\Lambda_{O,n}^a$. Thus, for all $\hat{x} \in \mathcal{B}_{T/2}^p$ which have a zeros in at least one of the overlapping sets $\Lambda_{O,n}^a$, the reconstruction described in the proof of Theorem 4.10 will fail. We are going to show that the set containig all such functions is in a sense small, namely a set of first category.

Lemma 4.11. *The set $\mathcal{G}$ of all $\hat{x} \in \mathcal{B}_{T/2}^p$ for which the reconstruction procedure of Theorem 4.10 fails is of first category.*

Proof. Let $\Lambda^a = \{\lambda_n\}_{n\in\mathbb{Z}}$ be a set of interpolation points as applied in the sampling scheme of Theorem 4.10, and set

$$\mathcal{G}_n := \{\hat{x} \in \mathcal{B}_{T/2}^p \;:\; \hat{x}(\lambda_n) = 0\} \quad \text{for } n \in \mathbb{Z}.$$

Let $\hat{y} \in \mathcal{B}_{T/2}^p$ be arbitrary but $\hat{y} \notin \mathcal{G}_n$, fix $n \in \mathbb{Z}$ and write $\lambda_n = \xi_n + \mathrm{i}\eta_n$. Then for every arbitrary $\hat{x} \in \mathcal{G}_n$, it follows from (3.46) that

$$|\hat{y}(\xi + \mathrm{i}\eta_n) - \hat{x}(\xi + \mathrm{i}\eta_n)| \leq C\,\|\hat{y} - \hat{x}\|_{\mathcal{B}_{T/2}^p}\,, \quad \text{for all } \xi \in \mathbb{R}$$

where C is a constant which depends only on p, T, and η_n but not on $\hat{x}$ and $\hat{y}$. For $\xi = \xi_n$ it follows in particular that $\|\hat{y} - \hat{x}\|_{\mathcal{B}_{T/2}^p} \geq \frac{1}{C}\,|\hat{y}(\lambda_n)| > 0$. This shows that there exists an open ball around every $\hat{y} \in \mathcal{B}_{T/2}^p$, $\hat{y} \notin \mathcal{G}_n$ which contains no element of $\mathcal{G}_n$. Thus $\mathcal{G}_n$ is nowhere dense in $\mathcal{B}_{T/2}^p$. Since $\mathcal{G} \subset \bigcup_{n\in\mathbb{Z}} \mathcal{G}_n$ and because the right hand side is the countable union of nowhere dense sets, $\mathcal{G}$ is of first category. □

From Lemma 4.11 we can conclude that the restriction on the signal space given in Theorem 4.10 is fairly mild. Intuitively, this follows also from the fact that the zeros of an entire function of exponential type can not be arbitrarily dense in $\mathbb{C}$. For example, by defining $\mathcal{Z}_n := \{z \in \mathbb{C} : n\pi/T < |z| \leq (n+1)\pi/T\}$, the result in [37] implies that for every $\hat{x} \in \mathcal{B}_{T/2}^p$ there exist only finitely many sets $\mathcal{Z}_n$ which contain more than one zero of $\hat{x}$. Consequently, by choosing the spacing of the interpolation

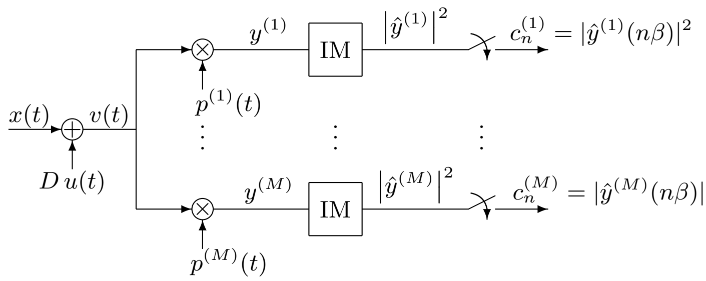


Figure 4.4: Measurement setup as in Fig. 4.2 but with an additional preprocessing of adding a test function $D\,u(t)$ prior to modulation, intensity measurement and sampling.

points in the overlapping sets $\Lambda^a_{O,n}$ to be less than π/T, it is very unlikely that a randomly chosen function from $\mathcal{B}^p_{T/2}$ fails to satisfy the condition of Theorem 4.10, especially for $a > 1$.

4.5 Signal Reconstruction in Subspaces

In this section we show that by minimal additional knowledge we can avoid even such pathological cases, so that all measured signals will definitely not have zeros in Λ^a. From the practical point of view, this can be achieved by adding a known test signal u to the desired signal $x \in \mathcal{Y}^p_{T/2}$ prior to the structured modulations (cf. Fig. 4.4), scaled according to its norm.

The addition of Du, with appropriate choice of the scale factor D, yields a function which only has zeros inside a strip parallel to the real axis. The following lemma, similar to a result by Duffin, Schaeffer [11], shows that this is the case when u is a sine-type function of type $T'/2 > T/2$.

Lemma 4.12. *Let $\hat{u}$ be sine-type function of type $T'/2$ and let $T \leq T'$ and $1 \leq p \leq \infty$ be arbitrary. For any $\hat{x} \in \mathcal{B}^p_{T/2}$ define the function*

$$\hat{v}(z) := \hat{x}(z) + D\,\hat{u}(z)\ .$$

Then $\hat{v} \in \mathcal{B}^\infty_{T'/2}$ and to every $\hat{u}$ and $D > 0$ there exists an $H > 0$ such that

$$\left|\hat{v}(\xi + \mathrm{i}\eta)\right| > 0 \quad \text{for all } |\eta| \geq H\ . \tag{4.27}$$

Proof. Since every sine-type function is bounded on $\mathbb{R}$ and because $\mathcal{B}^p_{T/2} \subset \mathcal{B}^p_{T'/2} \subset \mathcal{B}^\infty_{T'/2}$, it is immediately clear that $\hat{v} \in \mathcal{B}^\infty_{T'/2}$, and for all $z = \xi + \mathrm{i}\eta \in \mathbb{C}$ we have

$$\left|\hat{v}(z)\right| \geq \left|\,|D\,\hat{u}(z)| - |\hat{x}(z)|\,\right|\ . \tag{4.28}$$

Since $\hat{x} \in \mathcal{B}^p_{T/2}$ it follows from (3.46) that there is a constant $M = C \, \|\hat{x}\|$ such that

$$\left|\hat{x}(z)\right| \leq M \, \mathrm{e}^{\frac{T}{2}|\eta|} \, .$$

Similarly, by the definition of a sine-type function in Definition 3.20 one has the lower bound

$$\left|D \, \hat{u}(z)\right| \geq D \, A_u \, \mathrm{e}^{\frac{T'}{2}|\eta|} \, , \qquad \text{for all } |\eta| > H_u$$

with two constants A_u and H_u which depend on u and is therefore known. Therewith, (4.28) reads

$$\left|\hat{v}(z)\right| \geq D \, A_u \, \mathrm{e}^{\frac{T'}{2}|\eta|} - M \, \mathrm{e}^{\frac{T}{2}|\eta|} \, , \quad \forall \; |\eta| > H \, .$$

For $T = T'$, we can simply choose $D > \frac{M}{A_u}$ to obtain $|\hat{v}(z)| > 0$ for all $|\eta| > H = H_u$. In order to find a lower D we could apply oversampling, i.e. $T' > T$, so that $|\hat{v}(z)| > 0$ is achieved whenever $H = \max\left[H_u, \frac{2}{T'-T} \ln\left(\frac{M}{D\,A_u}\right)\right]$. Conversely for a given $H > H_u$, we can choose the constant D as

$$D > \frac{M}{A_u} \, \mathrm{e}^{\left(\frac{T}{2} - \frac{T'}{2}\right) H}$$

for (4.27) to hold.

□

Example 4.13. In [11], the authors choose $\hat{u}(z) = \cos(\frac{T'}{2} z)$. For this function, the constants A_u and H_u can be derived by

$$\begin{aligned} \left|\cos(\tfrac{T'}{2} z)\right| &= \frac{1}{2} \left| \mathrm{e}^{\mathrm{i}\frac{T'}{2}\xi} \, \mathrm{e}^{-\frac{T'}{2}\eta} + \mathrm{e}^{-\mathrm{i}\frac{T'}{2}\xi} \, \mathrm{e}^{\frac{T'}{2}\eta} \right| \\ &\geq \frac{1}{2} \left(\mathrm{e}^{\frac{T'}{2}|\eta|} - \mathrm{e}^{-\frac{T'}{2}|\eta|} \right) \geq \frac{1}{2} \left(\mathrm{e}^{\frac{T'}{2}|\eta|} - 1 \right) \\ &\geq \frac{1}{2} \left(1 - \mathrm{e}^{-\frac{T'}{2} H_u} \right) \mathrm{e}^{\frac{T'}{2}|\eta|} = A_u \, \mathrm{e}^{\frac{T'}{2}|\eta|} \qquad \forall |\eta| \geq H_u \end{aligned}$$

where $H_u > 0$ is arbitrary and $A_u = [1 - \exp(-\frac{T'}{2} H_u)]/2$. Note that the time domain signal corresponding to this sine-type function $\hat{u}$ is the non-regular tempered distribution $u = [\delta_{T'/2} + \delta_{-T'/2}]/2$ which vanishes on $\mathbb{T}$.

After adding a sine-type function $\hat{u}$ of type $T'/2$ to the desired signal $\hat{x} \in \mathcal{B}^p_{T/2}$, we obtain a function in $\hat{v} \in \mathcal{B}^\infty_{T'/2} \setminus C_0$. Therefore, by Condition B in Theorem 4.10 we will need oversampling $\tilde{T} > T' \geq T$ so that we can use the interpolation formula (4.17) choosing the kernel ψ_n as in (4.18).

Now we are ready to state a corollary of Theorem 4.10. By adding an appropriate test signal $D \, u(t)$ prior to our sampling scheme (cf. Fig. 4.4) we are able to ensure the "non-zero requirement" of Theorem 4.10, and therefore every signal in our signal space $\mathcal{Y}^p_{T/2}$ can be reconstructed from its magnitude measurement.

Corollary 4.14. *Consider a sampling scheme according to Sec. 4.3 which satisfies Condition A and B and with the additional pre-processing as shown in Fig.4.4. Then for every* $1 \leq p \leq \infty$ *there exists a function* $D\, u(t)$ *and an interpolation sequence* Λ^a *with overlap* $a \geq 1$ *such that every*

$$x \in \{x \in \mathcal{Y}^p_{T/2} \;:\; \|x\| \leq 1\} \tag{4.29}$$

can be perfectly reconstructed (up to a constant phase) from the measurements (4.20).

Remark. The additional assumption $\|x\| \leq 1$ only requires that an upper bound on the signal norm is known. Practically, this is necessary to calibrate the measurement system by an appropriate amplitude D of the additive test signal u.

Proof. Let x be from the set in (4.29). Using (3.46), it follows that there is a constant M such that $|\hat{x}(\xi)| \leq M$ for all $\xi \in \mathbb{R}$. Furthermore, without loss of generality we assume that $p = \infty$ since $\mathcal{B}^p_{T/2} \subset \mathcal{B}^\infty_{T/2}$ for all $p \in [1, \infty)$. Moreover let $\hat{u}$ be an arbitrary sine-type function of type $T'/2 \geq T/2$. Then it follows from Lemma 4.12 that there exist constants H, D such that the function

$$\hat{v}(z) := \hat{x}(z) + D\ \hat{u}(z) \tag{4.30}$$

has no zeros for all $z = \xi + \mathrm{i}\eta \in \mathbb{C}$ with $|\eta| > H$. Now choose $\Lambda^a = \{\lambda_k = \xi_k + \mathrm{i}\eta_k\}_{k\in\mathbb{Z}}$ as the zero set of a different sine-type function of type $\tilde{T}/2 > T'/2$. By Theorem 3.23 we can shift the imaginary parts of the interpolation points $\lambda_k = \xi_k + \mathrm{i}\eta_k$ such that $|\eta_k| > H$ for all k while Λ^a remains the zero set of a sine-type function. Denote the corresponding sine-type function by $S(z) = \lim_{n\to\infty} \prod_{|k|<n} \left(1 - \frac{z}{\lambda_k}\right)$.

Now the signal (4.30) is modulated and sampled exactly as described in Section 4.3. Then our intensity measurements are given, similar as in (4.20), by $c^{(m)}_n = |\langle \boldsymbol{\alpha}^{(m)}, \hat{\mathbf{v}}_n\rangle|^2$. Following the same steps as in the proof of Theorem 4.10, we obtain the values of $\hat{v}$ at the sampling set Λ^a up to a constant phase θ_0. Since by our construction, overlapping interpolation points cannot coincide with zeros of $\hat{v}$, the phase information can be propagated and we are able to recover $\hat{v}(\Lambda^a)\,\mathrm{e}^{\mathrm{i}\theta_0}$ from the intensity measurements for every signal $\hat{v}$ of the form (4.30).

For interpolation, we do not use Lemma 4.8 directly. Since $\hat{v} \in \mathcal{B}^\infty_{T'/2}$ and there exists an H such that $\sup_{k\in\mathbb{Z}} |\eta_k| < H < \infty$, it follows that the sequence $\{\hat{v}(\Lambda^a)\,\mathrm{e}^{\mathrm{i}\theta_0}\}$ is in ℓ^∞ such that Theorem 3.62 can be applied to interpolate $\hat{v}(z)\,\mathrm{e}^{\mathrm{i}\theta_0}$ from the sampling sequence $d_n := \hat{v}(\lambda_n)\,\mathrm{e}^{\mathrm{i}\theta_0}, n \in \mathbb{Z}$, using (3.80):

$$\hat{v}(z)\,\mathrm{e}^{\mathrm{i}\theta_0} = \sum_{n\in\mathbb{Z}} d_n \frac{S(z)}{S'(\lambda_n)} \left[\frac{1}{z - \lambda_n} + \frac{1}{\lambda_n}\right] .$$

But since θ_0 is unknown, it is not possible to determine $\hat{x}(z)$ directly from $\hat{v}(z)\,\mathrm{e}^{\mathrm{i}\theta_0}$ using (4.30). Instead, one determines

$$\begin{aligned} \tilde{x}(z) &:= \hat{v}(z)\,\mathrm{e}^{\mathrm{i}\theta_0} - D\,\hat{u}(z) \\ &= \hat{x}(z)\,\mathrm{e}^{\mathrm{i}\theta_0} - D\,\hat{u}(z)\left(1 - \mathrm{e}^{\mathrm{i}\theta_0}\right) . \end{aligned}$$

If we now choose $\hat{u}(z) = \cos(\frac{T'}{2} z)$ and apply the inverse Fourier-Laplace transform (2.32) to $\tilde{x}$ one obtains $\mathcal{F}^{-1}\tilde{x}(t) = x(t)\,\mathrm{e}^{\mathrm{i}\theta_0}$ for $t \in \mathbb{T}$ since the inverse Fourier transform of the cosine function vanishes on $\mathbb{T}$ (cf. Example 4.13). □

In this chapter we were able to show that perfect reconstruction for all signals in our signal space is possible only given magnitude measurement samples of the signal which is modulated in different ways. It is now interesting to look at the total sampling rate of this measurement setup and the robustness of this recovery method against noise, which we will both address in the next chapter.

5

Discussion and Outlook

5.1 Sampling rate

In order to find a sampling system in Fig. 4.2 which satisfies Condition A and B, one has to pick specific K, M, a and β. The number $K \geq 2$ can be chosen arbitrarily. Then $M = K^2$ is fixed, and $1 \leq a \leq K-1$. Given zeros $\{\lambda_k\}$, the sampling period β is determined by Condition B. As discussed before, one possible choice may start with the zeros of the function $\sin(\frac{\tilde{T}}{2}z)$ with $\tilde{T} \geq T$. Then $\delta := \lambda_k - \lambda_{k-1} = 2\pi/\tilde{T}$ such that $\beta = (K-a)\,\delta$. Therewith, the total sampling rate becomes

$$R(a,K,\tilde{T}) = \frac{M}{\beta} = \frac{K^2}{(K-a)\,\delta} = \frac{K^2}{K-a}\frac{\tilde{T}}{2\pi} = \frac{K^2}{K-a}\frac{\tilde{T}}{T}R_{\mathrm{Ny}}$$

where $R_{\mathrm{Ny}} := T/(2\pi)$ is the Nyquist rate. It is apparent that $R(a,K,\tilde{T})$ grows asymptotically proportional with K, increases with the overlap a, and is bounded below by

$$\inf_{\substack{1\leq a<K,\\ K\geq 1,\tilde{T}>T}} R(a,K,\tilde{T}) = \inf_{\tilde{T}>T} R(1,2,\tilde{T}) = 4R_{\mathrm{Ny}}\,.$$

Since $\tilde{T}/T$ can be made arbitrarily close to 1 using Theorem 4.10 and the proof of Corollary 4.14, we can sample at a rate which is almost as small as $4R_{\mathrm{Ny}}$ while still ensuring perfect reconstruction. This corresponds to the results in [2] for finite dimensional spaces, where it was shown that basically any $x \in \mathbb{C}^N$ can be reconstructed from $M \geq 4N-2$ magnitude samples.

We note that the above framework can be applied exactly the same way for bandlimited signals. To this end, one only has to exchange the time and frequency domain. Then the modulators in Fig. 4.2 have to be replaced by linear filters and the sampling of the magnitudes has to be done in the time domain.

5.2 Noise

In the previous chapter we assumed that the measurements are noiseless. However this is never the case in reality so that it is obligatory to discuss the robustness of our method against disturbances. Whenever we have noise in our samples, we will need to investigate its propagation through the three different stages of finite dimensional phase retrieval, phase propagation and interpolation. In the following, we will only consider additive noise so that the disturbed measurement is modelled as

$$\tilde{c_n}^{(m)} = |\langle a_n^{(m)}, x\rangle|^2 + w_n^{(m)} \text{ with } \sup_{\substack{n\in\mathbb{Z},\\ m=1\dots M}} |w_n^{(m)}| =: \epsilon_0 \tag{5.1}$$

Note that the disturbed matrix

$$\tilde{Q}_n = \frac{(K+1)}{K}\sum_{m=1}^{M} \tilde{c}_n^{(m)} Q_{\boldsymbol{\alpha}^{(m)}} - \frac{1}{K}\sum_{m=1}^{M}\tilde{c}_n^{(m)} I \tag{5.2}$$

is in general not rank-1 anymore and we have

$$\tilde{Q}_n - Q_n = \frac{K+1}{K}\sum_{m=1}^{M} w_n^{(m)} Q_{\boldsymbol{\alpha}^{(m)}} - \frac{1}{K}\sum_{m=1}^{M} w_n^{(m)} I \tag{5.3}$$

5.2.1 Effect of noise on Finite Dimensional Phase Retrieval

If we choose our $\alpha^{(m)}$ as a 2-uniform M/K-tight frame, the analytical formula (5.3) directly yields the Frobenius norm error which is

$$\begin{aligned}
\|\tilde{Q}_n - Q_n\|_2 &= \left\| \sum_{m=1}^{M} \frac{K+1}{K} w_n^{(m)} Q_{\alpha^{(m)}} - \frac{1}{K} w_n^{(m)} I \right\|_2 \\
&\le \sum_{m=1}^{M} w_n^{(m)} \left\| \frac{K+1}{K} Q_{\alpha^{(m)}} - \frac{1}{K} I \right\|_2 \\
&= \sum_{m=1}^{M} w_n^{(m)} \left\| \frac{K+1}{K} \Lambda_{Q_{\alpha^{(m)}}} - \frac{1}{K} I \right\|_2 \\
&= \sum_{m=1}^{M} w_n^{(m)} \sqrt{\left(\frac{K+1}{K}\lambda^{(m)} - \frac{1}{K}\right)^2 + (K-1)\frac{1}{K^2}} \\
&= \sum_{m=1}^{M} w_n^{(m)} \sqrt{\frac{K^2+K-1}{K^2}} \\
&\le \epsilon_0 K(K+1)
\end{aligned} \tag{5.4}$$

where U unitary and Λ_X is the diagonal matrix of the Hermitian matrix X with $U^*\Lambda_X U = X$ and $\lambda^{(m)}$ is the eigenvalue of the rank one matrix $Q_{\alpha^{(m)}}$. The third equation is obtained by noting that the Frobenius norm is the square-root of the Hilbert-Schmidt inner product and thus invariant under unitary transformations such as

$$\|A\|_2 = \|U^* A U\|_2$$

where U is unitary. The fourth equality follows from the representation of the Frobenius norm by the sum of squared singular values of the corresponding matrix (which are equivalent to the eigenvalues for Hermitian matrices). In general it is clear that $\lambda^{(m)} = \|\alpha^{(m)}\|_2^2$. We additionally know from [1] Definition 2.1. that $\|\alpha^{(m)}\|_2 = 1$ for all M so that the last equality holds. The inequality then follows from $M = K^2$.

The norm error of the matrix in (5.4) directly effects the error norm of its eigenvectors $\tilde{\hat{x}}_n - \hat{x}_n$. Herewith we use the following result.

Theorem 5.1. *Let $X = xx^*$ and $\|\tilde{X} - X\|_2 = \epsilon$, then*

$$\min_\phi \|\tilde{x} - \mathrm{e}^{\mathrm{i}\phi} x\|_2 \le \min(2\frac{\epsilon}{\|x\|_2}, 2\|x\|_2 + \sqrt{\epsilon}) \tag{5.5}$$

For the proof we need the following lemma.

Lemma 5.2. *For a minimally perturbed matrix $\tilde{A} := A + \delta A$ with a simple eigenvalue λ corresponding to column eigenvector v and row eigenvector w, we have that $\tilde{A}$ has the eigenvalue $\tilde{\lambda} = \lambda + \delta\lambda$ with*

$$\frac{\delta\lambda}{\lambda} \le \frac{\|w\|_2 \|v\|_2}{w^* v} \frac{\|\delta A\|_2}{\lambda} \tag{5.6}$$

where w is the row eigenvector of A. For rank-1 matrices we thus simply have

$$\delta\lambda \le \|\delta A\|_2 \, .$$

Proof. We use differentiation of the eigenvalue equality $Av = \lambda v$ to obtain

$$(\delta A)v + A(\delta v) = (\delta v)\lambda + v(\delta\lambda)$$

from which it follows that

$$w^*(\delta A)v = w^* v(\delta\lambda)$$

Using Cauchy-Schwartz the lemma is proved. □

Proof of Theorem 5.1. The proof is along the same lines as in [8].
For simplicity, lets denote the norm $a = \|x\|_2^2$. Without loss of generality we can assume that $x = \sqrt{a} e_1$ where $\{e_i\}_{i=1...K}$ is the canonical orthonormal basis. Then the eigenvalue of the rank-1 matrix X is $\lambda_1 = a$. Using Lemma 5.2 we can now state that

$$|\lambda_1 - \tilde{\lambda}_1| \le \|\tilde{X} - X\| = \epsilon$$

where $\tilde{\lambda_1}$ is the largest eigenvalue as $\tilde{X}$ might no more be a rank-1 matrix. It follows that

$$\tilde{\lambda}_1 \in [a-\epsilon, a+\epsilon]$$

and we can immediately state by the $\sin\theta$-Theorem [10] that

$$|\sin\theta| \leq \frac{\|\tilde{X}-X\|}{|\tilde{\lambda}_1|} \leq \frac{\epsilon}{a-\epsilon} \tag{5.7}$$

where $\theta \in [0, \frac{\pi}{2}]$ is the angle between the spaces spanned by the largest normed eigenvector of $\tilde{X}$, which is $\tilde{u}_1$, and e_1. This implies that we can write

$$\tilde{u}_1 = \cos\theta e_1 + \sin\theta e_1^{\perp} \tag{5.8}$$

where $e_1^{\perp}$ is the orthogonal vector to e_1. Now it naturally follows that

$$\|\sqrt{a}e_1 - \sqrt{\tilde{\lambda}_1}\tilde{u}_1\|_2^2 = (\sqrt{a} - \sqrt{\tilde{\lambda}_1}\cos\theta)^2 + \tilde{\lambda}_1 \sin^2\theta\,.$$

For small $\epsilon < \frac{a}{3}$ the following inequality holds

$$a+\epsilon \geq \sqrt{\tilde{\lambda}_1}\cos\theta \geq \sqrt{a-\epsilon-\frac{\epsilon^2}{a-\epsilon}} \geq \sqrt{a} - \frac{\epsilon}{\sqrt{a}} \tag{5.9}$$

which yields with (5.7)

$$\|\sqrt{a}e_1 - \sqrt{\tilde{\lambda}_1}\tilde{u}_1\|_2 = \sqrt{(\frac{\epsilon}{\sqrt{a}})^2 + \frac{(a+\epsilon)\epsilon}{(a-\epsilon)^2}} \leq \sqrt{\frac{\epsilon^2}{a} + \frac{4}{3}\frac{a\epsilon}{(\frac{2}{3})a^2}} = 2\frac{\epsilon}{\|x\|_2}\,.$$

At the same time we also have

$$\|\sqrt{a}e_1 - \sqrt{\tilde{\lambda}_1}\tilde{u}_1\|_2 \leq \sqrt{a} + \sqrt{a+\epsilon}\|\tilde{u}_1\|_2 \leq 2\sqrt{a+\epsilon} \leq 2(\|x\|_2 + \sqrt{\epsilon}) \tag{5.10}$$

from which the theorem follows. □

Remark. More results about the effect of matrix perturbations on their eigenvalues for example include the Bauer-Fike Theorem or the Weyl Theorems (see e.g. [19]).

By (5.4), we have for our phase retrieval problem that $\epsilon \leq \epsilon_0 K(K+1)$ given the supremum of the noise vector $\epsilon_0 = \sup_{n,m} |w_n^{(m)}|$ and thus

$$\min_{\phi} \|\tilde{x} - \mathrm{e}^{\mathrm{i}\phi}x\|_2 \leq \min\left(2\frac{\epsilon_0 K(K+1)}{\|x\|_2}, 2(\|x\|_2 + \sqrt{\epsilon_0 K(K+1)})\right)\,. \tag{5.11}$$

5.2.2 Error in phase propagation

Given the error estimate for the finite dimensional signal vector however does not allow us to model the disturbed vector which enters the phase propagation stage as an additive variation of the form

$$\tilde{\hat{x}}_n = \hat{x}_n + \delta_n \,. \tag{5.12}$$

Rather, it will most generally also contain a phase term for each element of the vector, i.e. $\tilde{\hat{x}}_n = (\hat{x}_n + \delta_n)\mathrm{e}^{\mathrm{i}\theta_n}$. Furthermore, the error at each step propagates through all the sampling points, which makes the problem difficult to analyse. However it is still interesting to consider the hypothetical case when $\theta_n \equiv 0$ and $\{\delta_n\} \in \ell^\infty$, i.e. (5.12) although it still needs to be discussed how $\|\{\delta_n\}\|_{\ell^\infty}$ relates to ϵ_0.

5.2.3 Error in the interpolation step

In the case $p \in (1, \infty)$ we know by Theorem 3.52 that if $\{\delta_n\} \in \ell^p$, then because of linearity of the interpolation formula, the error is represented by the series

$$\sum_{n\in\mathbb{Z}} \delta_n \hat{\psi}_n := \sum_{n\in\mathbb{Z}} \delta_n \frac{S(z)}{S'(\lambda_n)(z-\lambda_n)}$$

which converges in the $\mathcal{B}^p_\sigma$-norm, with $S(z)$ being sine-type functions of type σ. It remains to be seen whether there is a constant such that we can write

$$\|\sum_{n\in\mathbb{Z}} \delta_n \psi_n(z)\| \le C\|\{\delta_n\}\|_\infty \,. \tag{5.13}$$

For $p = \infty$ we know that the series in Theorem 3.62 does not converge for every bounded series $\{c_k\}_k \in \ell^\infty$. Therefore it is necessary to check under which conditions it does and whether one can find a bound of type (5.13). For the case of regular samples and exponential type $\sigma = \pi$, Levin has given a very nice result.

Theorem 5.3. *Let $\{c_k\} \in \ell^\infty$. Define the functional*

$$L_\tau : \{c_k\} \mapsto L_\tau(\{c_k\}) = \sum_{-\infty}^{\infty} (d_{k+\tau} - d_k)\frac{k}{k^2+1} \tag{5.14}$$

with $d_k = (-1)^k c_k$. In order that there exists an entire function $f(z)$ of exponential type $\le \pi$ which is bounded on the real axis and solves the interpolation problem $f(k) = c_k$, it is necessary and sufficient that

$$|L_\tau(\{c_k\})| < M, \quad \tau = 0, \pm 1, \pm 2, \ldots \tag{5.15}$$

for some $M > 0$.

Proof. A proof can be found in [22] Lecture 21 Theorem 2. Note first that the right hand side of (5.14) is always convergent for every τ. However, the important condition in this theorem is uniform boundedness of L_τ with respect to τ. The ultimate goal is to estimate

$$g(z) = \frac{\sin \pi z}{\pi} \sum_{n\in\mathbb{Z}}{}' (-1)^n c_n \left[\frac{1}{z-n} + \frac{1}{n}\right]. \tag{5.16}$$

The first step is to consider $|g(x+\tau+\mathrm{i})|$ for τ even and $0 \le x \le 2$ after which one can use Phragmen Lindelöf to obtain a bound for $|g(x)|$. The main idea is then to consider the difference for $0 \le x \le 2$

$$|g(x+\tau+\mathrm{i}) - g(x+\mathrm{i})| = \left|\frac{\sin \pi(x+\mathrm{i})}{\pi} \sum_{k\in\mathbb{Z}} \frac{d_{k+\tau}-d_k}{x-k+\mathrm{i}}\right| \tag{5.17}$$

$$\le \left|\frac{\sin \pi(x+\mathrm{i})}{\pi}\right| \left|\sum_{k\in\mathbb{Z}} \frac{d_{k+\tau}-d_k}{x-k+\mathrm{i}} - \frac{d_{k+\tau}-d_k}{k+\mathrm{i}}\right| \tag{5.18}$$

$$+ \left|\sum_{k\in\mathbb{Z}} \frac{d_{k+\tau}-d_k}{k+\mathrm{i}} - L_\tau(\{c_k\})\right| + |L_\tau(\{c_k\})| \tag{5.19}$$

$$= C_1 + C_2 + M \tag{5.20}$$

yielding

$$|g(x+\mathrm{i})| \le M + C_1 + C_2 + C_3 \tag{5.21}$$

with $C_3 = \max_{0\le x\le 2} |g(x+\mathrm{i})|$. Now the Phragmen Lindelöf Theorem 3.27 implies that $|g(z)|$ is bounded on the real axis which completes the proof. □

Remark. We can estimate some of the constants in the proof explicitly.

For C_2 we can use the following approximation with $\epsilon := \sup_k |c_k|$.

$$\begin{aligned}
\left|\sum_{k\in\mathbb{Z}} \frac{d_{k+\tau}-d_k}{x-k+\mathrm{i}} - L_\tau(\{c_k\})\right| &= \left|\sum_{k\in\mathbb{Z}} (d_{k+\tau}-d_k)\left(\frac{1}{k+\mathrm{i}} - \frac{k}{k^2+1}\right)\right| \\
&= \left|\sum_{k\in\mathbb{Z}} (d_{k+\tau}-d_k)\frac{1}{k^2+1}\right| \\
&\le |d_{k+\tau}-d_k| \sum_{k\in\mathbb{Z}} \frac{1}{k^2+1} \\
&\le \sup_{k\in\mathbb{Z}} 2|c_k|\left(1+2\frac{\pi^2}{6}\right)
\end{aligned}$$

For C_3 we have

$$\begin{aligned}|g(x+\mathrm{i})| &\leq \left|\frac{\sin(\pi(x+\mathrm{i}))}{\pi}\sum_{k\in\mathbb{Z}}(-1)^k c_k\left[\frac{1}{x+\mathrm{i}-k}+\frac{1}{k}\right]\right| \\ &\leq \frac{\sup|c_k|}{\pi}|x+\mathrm{i}|\sum_{k\in\mathbb{Z}}\left|\frac{1}{(x+\mathrm{i}-k)k}\right| \\ &\leq \frac{\epsilon\sqrt{5}}{\pi}\left(\sum_{k\in\mathbb{Z}}\frac{1}{k^2}\right)^{\frac{1}{2}}\left(\sum_{k\in\mathbb{Z}}\frac{1}{|x-k+\mathrm{i}|^2}\right)^{\frac{1}{2}} \leq \frac{\epsilon\sqrt{5}}{\pi}\left(1+2\frac{\pi^2}{6}\right)\end{aligned}$$

using the proof of Theorem 3.60 and where $|x+\mathrm{i}-k| \geq \delta = 1\ \forall x \in \mathbb{R}$. And thus it follows that

$$|g(x+\mathrm{i})| \leq M + C_1 + \epsilon(1+\frac{2\pi^2}{6})(2+\sqrt{5}\pi) \tag{5.22}$$

For C_1 an estimate remains to be found.

Since we have modelled our noise to fulfill (5.12), for the following series

$$\tilde{\hat{x}}(z) = \sum_{n\in\mathbb{Z}}(\hat{x}(\lambda_n)+\sigma_n)\frac{S(z)}{S'(\lambda_n)}\left[\frac{1}{z-\lambda_n}+\frac{1}{\lambda_n}\right]$$

we know that the error norm

$$\|\tilde{\hat{x}}-\hat{x}\|_\infty = \left\|\sum_{n\in\mathbb{Z}}\sigma_n\frac{S(\cdot)}{S'(\lambda_n)}\left[\frac{1}{\cdot-\lambda_n}+\frac{1}{\lambda_n}\right]\right\|_\infty$$

will converge if $\lambda_n = n$ and $\{\sigma_n\}$ satisfies the condition in Theorem 5.3. Whether we can find a constant C such that

$$\|\tilde{\hat{x}}(\omega)-\hat{x}(\omega)\|_\infty \leq C\sup_n|\sigma_n| \tag{5.23}$$

remains to be seen. It is also not clear whether a similar result can be obtained for irregular sampling, i.e. $\lambda_n \neq n$.

5.3 Summary and outlook

In this work we have shown that perfect phaseless reconstruction for continuous functions up to a constant phase is indeed possible with a sampling rate of four times the Nyquist rate. To this end we choose a particular measurement setup using multiple modulations of the signal - an approach which originates in heuristic methods in optics. It enables us to choose the measurement vectors of the finite dimensional phase retrieval subproblem such that it is solvable and to choose the sampling points such that there is a unique interpolating function. The Fourier domain of our signals are Bernstein spaces $\mathcal{B}^p_{T/2}$ and for regular distributions $x \in \mathcal{L}^{p'}(\mathbb{T})$ with $2 < p < \infty$ we actually have $\mathcal{L}^{p'}(\mathbb{T})$ convergence of the reconstruction series. In all other cases we achieve convergence in the topology of $\mathcal{S}'$. It was also shown that pathological cases where zeros of the signal in the Fourier domain coincide with the sampling points can be avoided. One possibility is by adding a known scaled sine-type function to the signal prior to modulation, where the scaling is determined by the maximal norm of the incoming signals which needs to be known a priori.

There are several remaining problems which are worth to be investigated in the future:

- How robust is this method against noise?
  - How does the error behave when noisy measurements reach the phase propagation step?
  - What happens if the noisy signal after the finite dimensional phase retrieval and phase propagation actually contains a multiplicative exponential noise term? These samples are used in the final interpolation step in Lemma 4.8. Thus, the simple additive model in (5.12) would not be valid anymore and the error needs to be analyzed in a different way than in Section 5.2.3.
  - Under which conditions does the series for $p = \infty$ in Section 5.2.3 converge for irregular sampling points? Can we find a bound as in (5.23)?
- Can we extend this framework to the two-dimensional case? This mainly involves interpolation of entire functions in two dimensions which is first of all non-trivial. However it is a crucial question if we were to apply it to real optical systems which generally deal with spatially limited two-dimensional signals.

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